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Appendix D

Inner Product Spaces

The inner product, taken of any two vectors in an arbitrary vector space, generalizes the dot product of two vectors in $\mathbb{R}^n$ *or* $\mathbb{C}^n$*.*

For two column vectors x and $y \in \mathbb{R}^n$ we can form two different vector products, namely

- the **outer product**

$$xy^T \; := \; \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \begin{pmatrix} y_1, ..., y_n \end{pmatrix} \; = \; \begin{pmatrix} x_1y_1 & \dots & x_1y_n \\ \vdots & & \vdots \\ x_ny_1 & \dots & x_ny_n \end{pmatrix} \; \in \; \mathbb{R}^{n,n}$$

and

- the standard **inner product**

$$x^Ty \; := \; \begin{pmatrix} x_1, ..., x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \; = \; x_1y_1 + ... + x_ny_n \; \in \; \mathbb{R} \; .$$

Here we interpret the two respective vectors as 1 by n or n by 1 matrices and multiply according to the rules of matrix multiplication. The outer product is a **dyadic product** since it creates an n by n **dyad** from two vectors, of which the first appears in column and the second in row form. It allows us to express **matrix multiplication** as a sum of rank 1 dyadic generators; see Sections 6.2, 10.2, and the proof of Lemma 3 in Section 12.2. A matrix product can be written as the sum of dyads of the columns and rows of the two matrix factors as follows:

$$A_{mn}B_{nk} = \begin{pmatrix} | & & | \\ a_1 & \dots & a_n \\ | & & | \end{pmatrix} \begin{pmatrix} - & b_1 & - \\ & \vdots & \\ - & b_k & - \end{pmatrix}$$
$$= \begin{pmatrix} | \\ a_1 \\ | \end{pmatrix} \begin{pmatrix} - & b_1 & - \end{pmatrix} + \ldots + \begin{pmatrix} | \\ a_n \\ | \end{pmatrix} \begin{pmatrix} - & b_k & - \end{pmatrix} \in \mathbb{R}^{m,k} ,$$

where the vectors a_i denote the columns of the first factor A and the b_j denote the rows of the second factor B.

The standard inner product $x^T y$ of two vectors in $\mathbb{R}^n$ is the same as the **dot product** $x \cdot y \in \mathbb{R}$ of the two vectors. It was introduced in Chapter 1 and interprets the first factor as a row and the second one as a column vector. The inner or dot product is also handy to express matrix multiplication, namely

$$A_{mn}B_{nk} = \begin{pmatrix} - & \widetilde{a_1} & - \\ & \vdots & \\ - & \widetilde{a_m} & - \end{pmatrix} \begin{pmatrix} | & & | \\ \widetilde{b_1} & \dots & \widetilde{b_k} \\ | & & | \end{pmatrix} = \begin{pmatrix} \widetilde{a_1} \cdot \widetilde{b_1} & \dots & \widetilde{a_1} \cdot \widetilde{b_k} \\ \vdots & & \vdots \\ \widetilde{a_m} \cdot \widetilde{b_1} & \dots & \widetilde{a_m} \cdot \widetilde{b_k} \end{pmatrix} \in \mathbb{R}^{m,k} ,$$

where the $\tilde{a}_i$ now denote the rows of A and the $\tilde{b}_j$ the columns of the second factor B. In addition, the inner or dot product helps define angles and orthogonality of two vectors in $\mathbb{R}^n$, see Chapters 10 through 12.

We start by listing four fundamental properties of the standard inner product of two vectors $.. \cdot .. : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Proposition 1: (Real Inner Product)

The standard inner product of two vectors x and $y \in \mathbb{R}^n$ is defined as

$$x \cdot y := x^T y = \begin{pmatrix} x_1, ..., x_n \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{i=1}^{n} x_i y_i \in \mathbb{R} .$$

It satisfies the following four properties:

(a) $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}^n$;

(b) $x \cdot (y + z) = x \cdot y + x \cdot y$ for all $x, y, z \in \mathbb{R}^n$;

(c) $(\alpha x) \cdot y = x \cdot (\alpha y) = \alpha(x \cdot y)$ for all $x, y \in \mathbb{R}^n$ and all $\alpha \in \mathbb{R}$.

(d) $x \cdot x \geq 0$ for all $x \in \mathbb{R}^n$, and $x \cdot x = 0 \in \mathbb{R}$ if and only if $x = 0 \in \mathbb{R}^n$. ◀

For two **complex vectors** $x, y \in \mathbb{C}^n$, several modifications are in order in the definition and the properties of an inner product due to the effects of complex conjugation, see Appendix A.

Proposition 2: (**Complex Inner Product**)

The standard inner product of two vectors x and $y \in \mathbb{C}^n$ is defined as

$$x \cdot y \ := \ x^* y \ = \ \left(\ \overline{x_1}, ..., \overline{x_n} \ \right) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \ = \ \sum_{i=1}^{n} \overline{x_i} y_i \ \in \ \mathbb{C} \ .$$

It satisfies the following four properties:

(a) $x \cdot y \ = \ \overline{y \cdot x}$ for all $x, y \in \mathbb{C}^n$;

(b) $x \cdot (y + z) \ = \ x \cdot y + x \cdot y$ for all $x, y, z \in \mathbb{C}^n$;

(c) $(\alpha x) \cdot y \ = \ x \cdot (\overline{\alpha} y) \ = \ \alpha (x \cdot y)$ for all $x, y \in \mathbb{C}^n$ and all $\alpha \in \mathbb{C}$.

(d) $x \cdot x \ \geq \ 0$ for all $x \in \mathbb{C}^n$, and $x \cdot x = 0 \in \mathbb{C}$ if and only if $x = 0 \in \mathbb{C}^n$. ◀

Proof: We deduce the four properties for the complex inner product only.

The properties of the real inner product in Proposition 1 follow immediately by dropping all complex conjugation bars in this proof.

(a) $x \cdot y \ = \ \sum_i \overline{x_i} y_i \ = \ \overline{\sum_i x_i \overline{y_i}} \ = \ \overline{\sum_i \overline{y_i} x_i} \ = \ \overline{y \cdot x}$ since double conjugation $\overline{\overline{c}}$ gives c back for any c in $\mathbb{C}$.

(b) $x \cdot (y + z) \ = \ \sum_i \overline{x_i}(y_i + z_i) \ = \ \sum_i \overline{x_i} y_i + \sum \overline{x_i} z_i \ = \ x \cdot y + x \cdot z$.

(c) $(\alpha x) \cdot y \ = \ \sum_i \overline{(\alpha x_i)} y_i \ = \ \overline{\alpha} \sum_i \overline{x_i} y_i \ = \ \overline{\alpha} x \cdot y$ and $\sum_i \overline{(\alpha x_i)} y_i \ = \ \sum_i \overline{x_i} (\overline{\alpha} y_i) \ = \ x \cdot (\overline{\alpha} y)$.

(d) $x \cdot x = \sum_i |x_i|^2 \geq 0$ as the sum of real squares. And equality holds precisely when $|x_i| = 0$ for each $i = 1, ..., n$, or when $x = 0 \in \mathbb{C}^n$. ■

The standard dot or inner product of $\mathbb{R}^n$ (or $\mathbb{C}^n$) serves very well in many aspects of linear algebra, such as when defining angles, orthogonality, and the length of vectors. More generally, an inner product can be defined in an arbitrary vector space V by requiring the four properties of a dot product; see Appendix C for abstract vector spaces.

Definition 1: Let V be an arbitrary vector space over a field of scalars $\mathbb{F}$.

(1) A function $\langle .., .. \rangle : V \times V \to \mathbb{F}$ that maps any two vectors f and $g \in V$ to the scalar $\langle f, g \rangle$ in $\mathbb{F}$ is **bilinear** if $\langle .., .. \rangle$ is linear in each of its arguments, i.e., if $\langle \alpha f + \beta g, h \rangle = \langle \alpha f, h \rangle + \langle \beta g, h \rangle$ and $\langle x, \delta u + \epsilon v \rangle = \langle x, \delta u \rangle + \langle x, \epsilon v \rangle$ for all scalars $\alpha, \beta, \delta, \epsilon \in \mathbb{F}$ and all vectors $x, u, v, f, g, h \in V$.

(2) A bilinear function $\langle .., .. \rangle : V \times V \to \mathbb{C}$ (or $\mathbb{R}$), operating on a complex (or real) vector space V, is an **inner product** on V if it satisfies the following four properties.

(a) $\langle x, y \rangle \ = \ \overline{\langle y, x \rangle}$ for all $x, y \in V$;

(b) $\langle x, (y + z) \rangle \ = \ \langle x, y \rangle + \langle x, y \rangle$ for all $x, y, z \in V$;

(c) $\langle (\alpha x), y \rangle \ = \ \langle x, (\overline{\alpha} y) \rangle \ = \ \alpha \langle x, y \rangle$ for all $x, y \in V$ and all $\alpha \in \mathbb{C}$.

(d) $\langle x, x\rangle \ \geq \ 0$ for all $x \in V$, and $\langle x, x\rangle = 0 \in \mathbb{C}$ if and only if $x = 0 \in V$. ◄

Note that if V is a vector space with the scalar field $\mathbb{R}$, then the complex conjugation in parts 2(a) and 2(c) above should simply be dropped.

For $V = \mathbb{R}^n$ and two vectors $x, y \in \mathbb{R}^n$, the standard dot product $\langle x, y\rangle := x \cdot y$ clearly defines an inner product on $\mathbb{R}^n$. We can express the dot product as $x \cdot y = x^T I y$ via the n by n identity matrix I. The matrix I_n is symmetric with n positive eigenvalues equal to 1 on its diagonal. Positive definite matrices generalize the properties of I; see Section 11.3. For example, every positive definite matrix $P = P^T \in \mathbb{R}^{n,n}$ can be expressed as a matrix product $P = A^T A$ with a nonsingular real square matrix A. For any $P = A^T A$ that is positive definite, we may set $\langle x, y\rangle_P := x^T P y = x^T A^T A y = (Ax)^T \cdot (Ay)$ and thereby obtain an inner product $\langle x, y\rangle_P := \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that differs from the ordinary dot product $x \cdot y = \langle x, y\rangle_I$. All one needs to do to verify this statement, is to show that $\langle .., ..\rangle_P$ is bilinear and satisfies the four standard properties of an inner product of Definition 1, see Problem 2 below.

Proposition 3: (a) Every bilinear form $f(x, y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be expressed as $f(x, y) = x^T S y$ for a real symmetric n by n matrix S.

(b) If $P = P^T \in \mathbb{R}^{n,n}$ is a positive definite real matrix, then $\langle x, y\rangle_P := x^T P y \in \mathbb{R}$ defines an inner product on $\mathbb{R}^n$.

(c) If $P = P^* \in \mathbb{C}^{n,n}$ is a positive definite complex matrix, then $\langle x, y\rangle_P := x^* P y \in \mathbb{C}$ defines an inner product on $\mathbb{C}^n$. ◄

Example 1: (a) To write $f(x, y) = 3x_1y_1 - 2x_1y_2 + 3x_3y_2 - x_2y_2 + 4x_4y_4 \ : \ \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ in the form $x^T S y$ with $S = S^T \in \mathbb{R}^{4,4}$, we define the diagonal entries s_{ii} of S as the coefficients of x_iy_i in f, or as $s_{11} = 3, \ s_{22} = -1, \ s_{33} = 0$, and $s_{44} = 4$. For $i > j$ we set the off-diagonal entries s_{ij} in S equal to half the coefficient of x_iy_j and then define $s_{ji} = s_{ij}$. Thus $s_{12} = -1 = s_{21}$ and $s_{32} = 1.5 = s_{23}$. Thus

$$S = \begin{pmatrix} 1 & -1 & & \\ -1 & 0 & 1.5 & \\ & 1.5 & -1 & \\ & & & 4 \end{pmatrix} = S^T.$$

(b) Determine whether $\langle x, y\rangle := x^T \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix} y$ is an inner product on $\mathbb{R}^2$.

Clearly $\langle x.y\rangle$ is bilinear in both x and $y \in \mathbb{R}^2$ since it is matrix generated. The generating matrix $S := \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix}$ with $\langle x.y\rangle = \langle x.y\rangle_S = x^T S y$ is symmetric, i.e., $S = S^T$, and hence its eigenvalues are real according to Section 11.1. Using the trace and determinant conditions of Theorem 9.5, we observe that the two eigenvalues of S add to 11 and multiply to $10 - 9 = 1$. Thus both eigenvalues of S must be positive real, making $S = S^T$ positive definite and $\langle x.y\rangle = \langle x.y\rangle_S$ an inner product on $\mathbb{R}^2$ according to Proposition 3.

(c) Determine whether $\langle u, v\rangle := u^T \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} v$ is an inner product on $\mathbb{C}^2$.

Clearly the function $\langle u.v\rangle$ maps any two complex 2-vectors to a complex number and it is bilinear. For $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X^*$ we observe that $X \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $X \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = -\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Therefore X has the eigenvalues 1 and -1 and X is not positive definite. Thus $\langle u, v\rangle = \langle u, v\rangle_X$ is not necessarily an inner product on $\mathbb{C}^n$ since Proposition 3 does not apply. In fact, $\langle u, v\rangle_X$ violates the fourth property (d) of inner products for $u = v = e_1 \neq 0 \in \mathbb{C}^2$:

$$\langle e_1, e_1\rangle_X = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0 \in \mathbb{C} .$$

Therefore $\langle u, v\rangle_X$ is not an inner product on $\mathbb{C}^2$. ◀

Inner products can help us measure and navigate in abstract vector spaces, such as in spaces of functions. As an example, we now consider the space of continuous functions $\mathcal{F}_{[0,1]} := \{f : [0, 1] \to \mathbb{R} \mid f \text{ continuous}\}$ defined on the interval $[0, 1] \subset \mathbb{R}$. This space is infinite dimensional, see Section 7.2(b). By setting $\langle f, g\rangle := \int_0^1 f(x)g(x)\, dx$ for all functions $f, g \in \mathcal{F}_{[0,1]}$, we have made $\mathcal{F}_{[0,1]}$ into an inner product space. Clearly $\langle f, g\rangle$ is linear in both of its function variables f and g since integration is linear in the sense that $\int u + v\, dx = \int u\, dx + \int v\, dx$. Identities such as

$$\langle \alpha u + \beta v, w\rangle = \int (\alpha u + \beta v)w\, dx = \alpha \int uw\, dx + \beta \int vw\, dx = \alpha\langle u, w\rangle + \beta\langle v, w\rangle$$

prove the properties (b) and (c) of an inner product. Next we observe that property (a) holds since $\langle f, g\rangle = \int fg\, dx = \int gf\, dx = \langle g, f\rangle$. And the first part of property (d) $\langle f, f\rangle = \int_0^1 f^2(x)\, dx \geq 0$ holds for any integrable function f since $f^2(x) \geq 0$. To show that $\langle f, f\rangle = 0$ for $f \in \mathcal{F}_{[0,1]}$ implies that $f = 0$ on $[0, 1]$ requires more thought: Every function $f \in \mathcal{F}_{[0,1]}$ is continuous. If f is not the zero function on $[0, 1]$, then $f(x_0) \neq 0$ for some $x_0 \in [0, 1]$. By continuity there is an interval $[a, b]$, $0 \leq a < b \leq 1$, with $x_0 \in [a, b]$ and $f(x) > \epsilon > 0$ for some given $\epsilon > 0$ and all $x \in [a, b]$. Using the additivity of the integral over its domain of integration, we observe that

$$\begin{aligned} \langle f, f\rangle &= \int_0^1 f^2(x)\, dx = \int_0^a f^2\, dx + \int_a^b f^2\, dx + \int_b^1 f^2\, dx \\ &\geq \int_a^b f^2\, dx \geq (b-a)\epsilon^2 > 0 . \end{aligned}$$

Consequently if f is continuous and $\langle f, f\rangle = 0$, then $f = 0$ on $[0, 1]$. Thus

$$\langle f,g\rangle \ := \ \int_0^1 f(x)g(x)\ dx$$

is an inner product on $\mathcal{F}_{[0,1]} = \{f : [0,1] \to \mathbb{R} \mid f \text{ continuous }\}$. Different inner product can be defined on $\mathcal{F}_{[0,1]}$ by setting

$$\langle f,g\rangle_w \ := \ \int_0^1 f(x)g(x)w(x)\ dx$$

for an arbitrary continuous **weight function** $w \in \mathcal{F}_{[0,1]}$ that is positive on $[0,1]$, see Problem 5.

Inner products define angles and orthogonality in abstract vector spaces V just as the standard dot products do in $\mathbb{R}^n$ and $\mathbb{C}^n$; recall Section 10.1. If both f and $g \neq 0 \in V$ and V is an inner product space with the inner product $\langle ..,..\rangle$, then the **angle** between f and g is defined with respect to a given inner product $\langle ..,..\rangle$ by the formula

$$\cos\angle(f,g) \ := \ \frac{\langle f,g\rangle}{\langle f,f\rangle^{\frac{1}{2}}\langle g,g\rangle^{\frac{1}{2}}}\ .$$

And $f \in V$ is **orthogonal** to $g \in V$, or $f \perp g \in V$, if $\langle f,g\rangle = 0$ for the inner product $\langle ..,..\rangle$ of V. Here $\langle f,g\rangle$ may be complex for two functions f and g when V is a complex vector space. In this case the complex valued cosine function $\cos(z)$ is used.

Example 2: (a) In $V = \mathcal{F}_{[0,1]}$ with the inner product $\langle f,g\rangle := \int_0^1 2f(x)g(x)\ dx$, find the angle between the two functions $f(x) = 1$ and $g(x) = x \in V$.
To find the angle we have to evaluate three different inner products: $\langle f,g\rangle = \int_0^1 2x\ dx = x^2\ \big|_0^1 = 1$, $\langle f,f\rangle = \int_0^1 2\ dx = 2x\ \big|_0^1 = 2$, and $\langle g,g\rangle = \int_0^1 2x^2\ dx = \frac{2x^3}{3}\ \big|_0^1 = \frac{2}{3}$. Therefore $\cos\angle(f,g) = \dfrac{\langle f,g\rangle}{\langle f,f\rangle^{\frac{1}{2}}\langle g,g\rangle^{\frac{1}{2}}} = \dfrac{1}{\sqrt{2}\sqrt{\frac{2}{3}}} = \dfrac{\sqrt{3}}{2}$. And the angle between f and g has the radian measure of $\arccos\left(\frac{\sqrt{3}}{2}\right)$. Note that the inner product contains the **weight function** $w(x) = 2$.

(b) Show that the two functions $h(x) = 1$ and $k(x) = x - \frac{1}{2}$ are orthogonal in V of part (a) with its given inner product.
We evaluate $\langle h,k\rangle = \int_0^1 2(x-\frac{1}{2})\ dx = \int_0^1 2x\ dx - \int_0^1\ dx = x^2\ \big|_0^1\ -x\ \big|_0^1 = 1-1 = 0.$

(c) Find an orthogonal basis for the subspace $\text{span}\{x,x^2\} \subset \mathcal{F}_{[0,1]}$ with respect to the inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)\ dx$.

Here we use the modified Gram–Schmidt process of Chapter 10 for the functions $u_1 = x$ and $u_2 = x^2$.

$$\begin{aligned} v_1 &:= u_1 = x \ ; \\ v_2 &:= \langle v_1, v_1\rangle u_2 - \langle u_2, v_1\rangle v_1 \ . \end{aligned}$$

We have $\langle v_1, v_1\rangle = \int_0^1 x^2 \ dx = \frac{x^3}{3} \ |_0^1 = \frac{1}{3}$ and $\langle u_2, v_1\rangle = \int_0^1 x^3 \ dx = \frac{x^4}{4} \ |_0^1 = \frac{1}{4}$. Thus $v_1 = x$ and $v_2 = \frac{1}{3}u_2 - \frac{1}{4}v_1 = \frac{1}{3}x^2 - \frac{1}{4}x$ are orthogonal in $\mathcal{F}_{[0,1]}$ with respect to the particular inner product. Normalizing the v_i with respect to the given inner product $\langle .., ..\rangle$ makes $w_1 = \dfrac{1}{\sqrt{\langle v_1, v_1\rangle}} \, v_1 = \sqrt{3}\, x$ and $w_2 = \dfrac{1}{\sqrt{\langle v_2, v_2\rangle}} \, v_2 = 4\sqrt{5}\ x^2 - 3\sqrt{5}\ x$ since $\langle v_1, v_1\rangle = 1/3$ and $\langle v_2, v_2\rangle = 1/(12^2 \cdot 5)$. The students should check this assertion by evaluating $\langle v_1, v_2\rangle = \int_0^1 v_1(x)v_2(x) \ dx$, $\langle v_1, v_1\rangle$, and $\langle v_2, v_2\rangle$. ◀

Inner products define vector norms in a natural way.

Proposition 4: If $\langle .., ..\rangle$ is an inner product on a real vector space V, then

$$\|x\|_{\langle .., ..\rangle} := \langle x, x\rangle^{1/2} : V \to \mathbb{R}$$

defines a **vector norm** for every $x \in V$ with the following properties:

(1) $\|x\|_{\langle .., ..\rangle} \geq 0$ for all $x \in V$, and $\|x\|_{\langle .., ..\rangle} = 0 \in \mathbb{R}$ if and only if $x = 0 \in V$.

(2) $\|\alpha x\|_{\langle .., ..\rangle} = |\alpha| \|x\|_{\langle .., ..\rangle}$ for all vectors $x \in V$ and all scalars $\alpha \in \mathbb{R}$.

(3) $|\langle x, y\rangle| \leq \|x\|_{\langle .., ..\rangle} \|y\|_{\langle .., ..\rangle}$ for all vectors $x, y \in V$. (**Cauchy–Schwarz inequality**)

(4) $\|x + y\|_{\langle .., ..\rangle} \leq \|x\|_{\langle .., ..\rangle} + \|y\|_{\langle .., ..\rangle}$ for all vectors $x, y \in V$. (**triangle inequality**)

◀

A vector norm $\|..\|$ is called **induced** by the inner product $\langle .., ..\rangle$ if $\|..\| = \langle .., ..\rangle^{1/2}$ as it is in Proposition 4. General vector norms without an underlying inner product can, however, be solely defined by the three norm defining properties (1), (2), and (4) of Proposition 4. Proofs of both the Cauchy–Schwarz and the triangle inequality for the standard **euclidean norm** $\|x\| = \sqrt{x^*x}$ of $\mathbb{R}^n$ or $\mathbb{C}^n$ are outlined in Section 10.1 and Problems 23 and 26 in Section 10.1.P.

Definition 2: A function $g(x) : V \to \mathbb{R}$ is a **vector norm** on a real vector space V if it satisfies the properties (1), (2), and (4) of Proposition 4.

Example 3: The function $g(x) := \max_{i=1}^{n}\{|x_i|\} : \mathbb{R}^n \to \mathbb{R}$ is a vector norm on $\mathbb{R}^n$.

To see this we check the three conditions of a vector norm: Clearly $g(x) \geq 0$ for all $x \in \mathbb{R}^n$, and $g(x) = 0$ if and only if $\max|x_i| = 0 \in \mathbb{R}$, or if and only if $x = 0 \in \mathbb{R}^n$, establishing property (1). Next $g(x)$ satisfies property (2) since $g(\alpha x) = \max\{|\alpha x_i|\} = \max\{|\alpha||x_i|\} = |\alpha|\max\{|x_i|\} = |\alpha| g(x)$. Finally, the triangle inequality $|\alpha + \beta| \leq |\alpha| + |\beta|$ for scalars $\alpha, \beta \in \mathbb{R}$ helps us prove property (4):

$$g(x+y) = \max\{|x_i + y_i|\} \leq \max\{|x_i| + |y_i|\} \leq \max\{|x_i|\} + \max\{|y_i|\} = g(x) + g(y) \ .$$

Thus $g(x)$ is a vector norm.

In Example 5 we learn that g is not induced by any inner product of $\mathbb{R}^n$. ◀

A vector norm $\|..\|$ measures the length of vectors in V, just as the standard euclidean norm $\|u\| := \sqrt{u^T u}$ measures the length of vectors in $\mathbb{R}^n$ via the standard dot product.

Example 4: (a) Find the length of the standard unit vector $e_1 \in \mathbb{R}^2$ in terms of the vector norm that is induced by the inner product $\langle x, y\rangle = x^T \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix} y$ of Example 1(a).

We compute

$$\|e_1\|^2_{\langle ..,..\rangle} = \langle e_1, e_1\rangle_{\langle ..,..\rangle} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 10 & 3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 10 \\ 3 \end{pmatrix} = 10 \, .$$

Therefore $\|e_1\|_{\langle ..,..\rangle} = \sqrt{10}$.

(b) Find the norm of the function $f(x) = x^2$ in the function space V of Example 2(a).

We have $\langle f, f\rangle = \int_0^1 2x^4 \, dx = \frac{2x^5}{5} \Big|_0^1 = \frac{2}{5}$, giving f the length, or norm $\|f\|_{\langle ..,..\rangle} = \langle f, f\rangle^{1/2} = \sqrt{\frac{2}{5}}$.

(c) Find the length of the vector $w = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ for the norm that is induced by the inner product $\langle x, y\rangle := x^T \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} y$ on $\mathbb{R}^3$.

Clearly the matrix $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ is positive definite as a positive diagonal matrix. Therefore $\langle ..,..\rangle$ is an inner product according to Proposition 3. Next we compute the induced vector norm of w:

$$\|w\|^2_{\langle..,..\rangle} = \langle w, w\rangle = \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 6 \end{pmatrix} = 2+1+12 = 15\ ,$$

or $\|w\|_{\langle..,..\rangle} = \sqrt{15}$. Note that in the euclidean norm $\|w\|_2 := \sqrt{w^T I w} = \sqrt{w^T w} = \sqrt{6}$.

(d) Show that for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2$, the function $f(x) = \sqrt{3x_1^2 + 4x_1x_2 + 3x_2^2} : \mathbb{R}^2 \to \mathbb{R}$ is an induced vector norm.

We have $f^2(x) = 3x_1^2 + 4x_1x_2 + 3x_2^2 = \begin{pmatrix} x_1 & x_2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x^T \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} x$; see Example 6 in Section 11.3 for more on quadratic forms such as f. The matrix $A := \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} = A^T$ is real symmetric with eigenvalues that sum to its trace 6 and that multiply to its determinant $9 - 4 = 5$, according to Theorem 9.5. Thus the eigenvalues of A are 5 and 1, making A positive definite and $\langle x, y\rangle := x^T \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix} y$ an inner product on $\mathbb{R}^2$ due to Proposition 3. This inner product induces $f(x)$ as a norm on $\mathbb{R}^2$. ◀

All inner products $\|..\|_{\langle..,..\rangle}$ that are induced by an inner product $\langle .., ..\rangle$ satisfy the **parallelogram identity**.

Proposition 5: If $\|..\|_{\langle..,..\rangle}$ is the induced vector norm for the inner product $\langle .., ..\rangle$ of an arbitrary real vector space V, then the **parallelogram identity**

$$\frac{1}{2}\left(\|x+y\|^2_{\langle..,..\rangle} + \|x-y\|^2_{\langle..,..\rangle}\right) = \|x\|^2_{\langle..,..\rangle} + \|y\|^2_{\langle..,..\rangle}$$

holds for all $x, y \in V$. ◀

The parallelogram identity states that the sum of the squared lengths of the two sides x and y of any parallelogram equals the average of the lengths of its two diagonals $x+y$ and $x-y$ squared for any induced vector norm.

Proof: To prove the parallelogram identity in a real vector space we expand its left hand side by using the properties of the norm inducing inner product.

$$\begin{aligned}
\frac{1}{2}\left(\|x+y\|^2_{\langle..,..\rangle} + \|x-y\|^2_{\langle..,..\rangle}\right) &= \frac{1}{2}\left(\langle x+y, x+y\rangle + \langle x-y, x-y\rangle\right) \\
&= \frac{1}{2}\langle x,x\rangle + \langle x,y\rangle + \frac{1}{2}\langle y,y\rangle + \frac{1}{2}\langle x,x\rangle - \langle x,y\rangle + \frac{1}{2}\langle y,y\rangle \\
&= \langle x,x\rangle + \langle y,y\rangle \;=\; \|x\|^2_{\langle..,..\rangle} + \|y\|^2_{\langle..,..\rangle} \;.
\end{aligned}$$

■

Example 5: The **maximum vector norm** $\|x\|_\infty := \max_{i=1}^{n}\{|x_i|\}$ of $\mathbb{R}^n$ from Example 3 is not induced by any inner product of $\mathbb{R}^n$ since it does not satisfy the parallelogram identity.

For example, in $\mathbb{R}^2$ we have for $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ that $\|x\|_\infty = \|y\|_\infty = 1 = \|x-y\|_\infty$ and $\|x+y\|_\infty = 2$ since $x+y = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $x-y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Thus

$$\frac{1}{2}\left(\|x+y\|^2_\infty + \|x-y\|^2_\infty\right) = \frac{1}{2}(4+1) = \frac{5}{2} \;\neq\; 2 = 1+1 = \|x\|^2_\infty + \|y\|^2_\infty \;.$$

◀

Proposition 4 makes every inner product space a normed vector space. However, in Example 5 and more generally in *functional analysis*, it has been shown that not all vector norms derive from inner products. To complete our elementary explorations of inner product spaces and normed vector spaces, we mention without proof that all normed vector spaces whose norm $\|..\|$ satisfies the parallelogram identity of Proposition 5 can be made into an inner product space by setting

$$\langle x, y\rangle \;:=\; \frac{\|x+y\|^2 - \|x\|^2 - \|y\|^2}{2} \;.$$

Moreover, the parallelogram identity can be generalized to complex vector spaces, but this is beyond the scope of this appendix and elementary linear algebra.

A.D.P Problems

1. Show that the function $f(x,y) = 2x_1y_1 - x_2y_2 + 4x_3y_3 + x_2y_3 : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is a bilinear function for $x, y \in \mathbb{R}^3$. Is this function an inner product on $\mathbb{R}^3$? Does f induce a norm on $\mathbb{R}^3$?

2. (a) Show that $\langle x, y\rangle_A := x^TAy$ is a bilinear function for every real n by n matrix A.

   (b) Show that if $P = P^T \in \mathbb{R}^{n,n}$ is positive definite, then P can be expressed as $P = A^TA$ for some nonsingular real matrix A.
   (Hint: Use Chapter 11: Diagonalize P orthogonally as $U^TPU = D = \sqrt{D}\sqrt{D}$ for a positive diagonal matrix D and extract P from this matrix equation.)

   (c) Show: If $P = P^T \in \mathbb{R}^{n,n}$ is positive definite, then $\langle x, y\rangle_P := x^TPy$ is an inner product on $\mathbb{R}^n$.
   (Hint: Use part (b).)

3. Test whether the following functions are (a) bilinear and (b) inner products on their respective spaces:

(1) $h(x,y) = x^T \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4 & 2 \\ 0 & 1 & 1 \end{pmatrix} y$ for $x, y \in \mathbb{R}^3$.

(2) $k(x,y) = x^T \begin{pmatrix} 4 & 2 \\ 1 & 1 \end{pmatrix} y$ for $x, y \in \mathbb{R}^2$.

(3) $\ell(x,y) = x^T \begin{pmatrix} 4 & 2 \\ 1 & -1 \end{pmatrix} y$ for $x, y \in \mathbb{R}^2$.

(4) $m(x,y) = x^T \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix} y$ for $x, y \in \mathbb{R}^2$.

(5) $n(x,y) = 2x_1^2 - 3x_1y_1 + 4y_1^2$ on $\mathbb{R}^2$.

(6) $p(x,y) = -x_1y_2$ on $\mathbb{R}^2$.

4. (a) Find the length of the two vectors $x = \begin{pmatrix} 1 & \dots & 1 \end{pmatrix}$ and $y = \begin{pmatrix} 1 & 0 & \dots & 0 & -1 \end{pmatrix} \in \mathbb{R}^n$ for both the standard euclidean vector norm and for the maximum vector norm.

(b) Find the cosine of the angle between x and $y \in \mathbb{R}^n$ in part (a) for

(1) the standard inner product and the

(2) inner product $x^T \begin{pmatrix} 3 & 1 & & 0 \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & 1 & 3 \end{pmatrix} y$

of $\mathbb{R}^n$.

5. If $w(x) > 0$ is continuous on the interval $[0,1]$, prove that $\langle f,g\rangle_w := \int_0^1 f(x)g(x)w(x)\ dx$ is an inner product on the space of continuous functions $\mathcal{F}_{[0,1]}$.

6. Construct a positive and continuous weight function $w(x)$ so that the two functions $f(x) = 1$ and $g(x) = x \in \mathcal{F}_{[0,1]}$ become orthogonal with respect to the inner product $\langle f,g\rangle_w := \int_0^1 f(x)g(x)w(x)\ dx$, if possible.

7. Construct a positive and continuous weight function $w(x)$ so that the two functions $f(x) = 1$ and $g(x) = x - \frac{1}{2} \in \mathcal{F}_{[0,1]}$ are not orthogonal with respect to the inner product $\langle f,g\rangle_w := \int_0^1 f(x)g(x)w(x)\ dx$, if possible.

8. Orthogonalize the two functions $f(x) = 1$ and $g(x) = x \in \mathcal{F}_{[0,1]}$ with respect to the weighted inner product $\langle f,g\rangle_w := \int_0^1 f(x)g(x)x^2\ dx$.

9. Orthogonalize the three functions $f(x) = 1$, $g(x) = x$, and $h(x) = x^2$ in $\mathcal{F}_{[0,1]}$ with respect to the inner product $\langle f,g\rangle := \int_0^1 f(x)g(x)\ dx$.

10. Show that the standard euclidean vector norm $\|x\| := \sqrt{x^Tx}$ for $x \in \mathbb{R}^n$ satisfies the parallelogram identity.

11. Show that $\|x\|_1 := \sum_{i=1}^n |x_i|$ is a vector norm on $\mathbb{R}^n$. Is it induced by an inner product or not?

12. Examine whether the vector norm $\|x\|_1$ of $\mathbb{R}^n$ in the previous problem is an induced vector norm.

13. (a) Assume that the norm $\|..\|$ is an induced norm on V. If $\|u\| = 4$, $\|u+v\| = 6$, and $\|u-v\| = 5$, what is the length of v? What is the distance between u and v?

(b) Repeat part (a) for $\|u\| = 7$.

14. Let V be a real inner product space. Let $\|..\|$ be the vector norm that is induced by the inner product $\langle ..,..\rangle$ on V. Show that

$$\langle x,y\rangle = \frac{\|x+y\|^2 - \|x-y\|^2}{4}$$

holds for all $x, y \in V$.

15. Let $\{u_1, ..., u_k\}$ be an orthonormal set of vectors in an inner product space of finite dimension $n \geq k$. Prove that $x \in \text{span}\{u_1,...,u_k\}$ if and only if $\|x\|^2 = |\langle x,u_1\rangle|^2 + \ ...\ + |\langle x,u_k\rangle|^2$ for the induced vector norm.

16. (a) Does $\langle x,y\rangle := x^* \begin{pmatrix} 2 & -i \\ i & 1 \end{pmatrix} y$ define an inner product on $\mathbb{C}^2$?

(b) Repeat part (a) for $\langle x,y\rangle := x^* \begin{pmatrix} i & i \\ i & 1 \end{pmatrix} y$.

(c) Repeat part (a) for $\langle x,y\rangle := \overline{x_1}y_1 - 2\overline{x_2}y_2$.

(d) Are any of the functions in parts (a) through (c) bilinear?

17. In a normed vector space V we define the **distance function** between any two vectors as $d(x, y) := \|x - y\|$. Show that

(a) $d(x, y) = d(y, x)$,

(b) $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, and $z \in V$.

(c) Evaluate the distance between $x = \begin{pmatrix} 1 & -2 & 4 & -3 \end{pmatrix}$ and $y = \begin{pmatrix} 0 & 2 & -1 & -3 \end{pmatrix} \in \mathbb{R}^4$ for the euclidean norm and the maximum norm.

18. (a) Assume that $\langle x, y\rangle_H := x^T H x$ and $\langle x, y\rangle_K := x^T K x$ are two bilinear forms on $\mathbb{R}^n$ with H and K both n by n and positive definite.
If $\langle x, y\rangle_H = \langle x, y\rangle_K$ for all $x, y \in \mathbb{R}^n$, show that $H = K$ as matrices.

(b) Show: If $x^T A y = 0$ for a real symmetric matrix A and all vectors $x, y \in \mathbb{R}^n$, then $A = O_n$.

19. Let $x = \begin{pmatrix} 2-i & 1+i & 3 \end{pmatrix}$ and $y = \begin{pmatrix} i & 2 & -i \end{pmatrix} \in \mathbb{C}^3$. For the standard euclidean vector norm of $\mathbb{C}^3$, compute

(a) $\|x\|$ and $\|y\|$, (b) the distance $d(x, y)$,
(c) $\|y - x\|$, (d) $\|x + y\|^2$, and
(e) the angle between x and $y \in \mathbb{C}^3$.

20. In $\mathcal{F}_{[0,1]}$ with the standard inner product $\langle f, g\rangle := \int_0^1 f(x)g(x)\, dx$, determine the 'length' of the two functions $f(x) = 3x - 2$ and $g(x) = x^2 + x \in \mathcal{F}_{[0,1]}$, as well as the cosine of the angle between f and g.

21. Consider the function $f(p, q) : \mathcal{P}_n \times \mathcal{P}_n \to \mathbb{R}$ defined by $f(p, q) = p(1)q(1) + 2p(2)q(2) + 3p(3)q(3)$, where $\mathcal{P}_n$ denotes the real variable polynomials of degree not exceeding n.

(a) Show that f is bilinear on $\mathcal{P}_n \times \mathcal{P}_n$.

(b) Show that f induces a norm $\|..\|$ on $\mathcal{P}_m$ for all $m \leq 2$.

(c) Show that f does not induce a norm on any $\mathcal{P}_m$ if $m > 2$.

(d) Find $\|2x^2 - 3x\|$ and $\|1 - x^2 - x^3 + 4x\|$ in the vector norm that is induced by f.

(e) Write down a formula for the induced norm $\|p\|$ and $p \in \mathcal{P}_2$.

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