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Substitution Elasticities with Many Inputs

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Abstract—Hicks-Allen elasticities of substitution describe the local convexity of isoquants when there are two inputs. This note reviews various approaches to reporting elasticities of substitution for pairs of inputs when there are many inputs. A bilateral elasticity is introduced which holds other inputs constant and performs better than Morishima and McFadden elasticities.

Keywords—Substitution, Elasticity, Bilateral, Cross price, Complementarity.

1. INTRODUCTION

Hicks [1] emphasizes that the pattern of income redistribution between two inputs due to a change in their relative price depends upon their substitution elasticity. Allen [2] formalizes this notion and demonstrates that with two inputs the Hicks substitution elasticity equals the Allen partial elasticity defined in terms of the production function.

When there are many (more than two) inputs, Morishima [3] elasticities and McFadden [4] elasticities are regularly reported for pairs of inputs, as in [5]. There may in fact arise circumstances with many inputs when only isolated input changes would matter. For instance, some inputs or their prices could become fixed by contract, and profit or cost could be regulated. Blackorby and Russell [6] revive interest in substitution elasticities with many inputs.

Hicks-Allen substitution elasticities are generally a function of the percentage changes in factor prices as well as relevant cross price elasticities. As a result, a summary elasticity meant to apply for any percentage change in factor prices is inherently distorting. The Morishima [3] elasticity allows only one input price to change, and Mundlack [7] classifies it as a two input, one price elasticity. The McFadden [4] elasticity is a shadow elasticity which holds cost constant.

This note derives a bilateral substitution elasticity which holds other inputs constant but allows both for changes in relative input prices and for cost minimization. Inputs thus adjust to a particular relative input price change, exactly as with two inputs. Its performance in applications is compared with Morishima and McFadden elasticities.

2. HICKS-ALLEN, ALLEN PARTIAL, AND CROSS PRICE ELASTICITIES

The Hicks-Allen elasticity between factors i and k is the percentage change in the ratio of inputs due to a 1% change in the ratio of their prices. The Hicks-Allen elasticity of substitution

R. Saba, C. Blackorby and O. Ashenfelter provided comments. The usual caveat applies.

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is written

$$\alpha_{ik} \equiv \frac{d\ln(a_i/a_k)}{d\ln(w_k/w_i)}. \tag{1}$$

The underlying production function is written $q = f(a)$, where a is the vector of factor inputs. With constant returns to scale, Allen [2, Section 13.7] argues that

$$\alpha_{ik} = \frac{f_i f_k}{q f_{ik}},$$

where subscripts denote partial derivatives of $f(a)$.

Allen [2, Section 19.5] subsequently defines the Allen partial elasticity of substitution,

$$\pi_{ik} \equiv \frac{qF_{ik}}{a_i a_k F}, \tag{2}$$

where F is the determinant of the bordered Hessian matrix of partials and cross partials of the production function, and F_{ik} refers to the cofactor of element i, k.

The cross price elasticity between the price of factor k and input of factor i is written

$$\varepsilon_{ik} \equiv \frac{\hat{a}_i}{\hat{w}_k}, \tag{3}$$

where $\hat{}$ represents percentage change. The cross price elasticity is shown to be equal to $\theta_k \pi_{ik}$, where θ_k is the share of factor k in the cost of production. Allen points out that the Hicks-Allen elasticity of substitution α_{ik} in the many input case is related to the cross price elasticity ε_{ik}, but does not explore the relationship.

3. MORISHIMA AND MCFADDEN ELASTICITIES

The Morishima [3] elasticity between inputs i and k in a production function with many inputs, developed independently by Blackorby and Russell [8], is written

$$\mu_{ik} \equiv \frac{d\ln(a_i/a_k)}{d\ln w_k}. \tag{4}$$

Relative input price changes are not explicitly considered in the Morishima elasticity. In terms of cross price elasticities,

$$\mu_{ik} = \frac{(\hat{a}_i - \hat{a}_k)}{\hat{w}_k} = \varepsilon_{ik} - \varepsilon_{kk}. \tag{5}$$

As discussed by Blackorby and Russell [6], μ_{ik} is constant with a constant elasticity of substitution (CES) production function. In computations, μ_{ik} will be positive and much larger than ε_{ik}, since the own price elasticity ε_{kk} is negative and relatively large in magnitude.

The McFadden elasticity allows for change in the relative input price, but holds cost constant. Cost equals the sum of payments to factors, $C = \sum_j w_j a_j$. Differentiating and using a cost minimizing envelope result, $dC = \sum_j a_j dw_j$. Suppose the only input prices which change are w_i and w_k. With cost constant,

$$0 = \hat{C} = \theta_i \, \hat{w}_i + \theta_k \, \hat{w}_k. \tag{6}$$

Since $\hat{w}_k/\hat{w}_i = -\theta_i/\theta_k$, the percentage change in relative inputs can be written

$$d\ln\left(\frac{a_i}{a_k}\right) = \hat{a}_i - \hat{a}_k = 0.5(m_{ik}\hat{w}_k - \mu_{ki}\hat{w}_i) = -.5\hat{w}_i\left(\mu_{ki} + \left(\frac{\theta_i}{\theta_k}\right)\mu_{ik}\right). \tag{7}$$

From (6), the percentage change in the relative input price is

$$d\ln\left(\frac{w_k}{w_i}\right) = \hat{w}_k - \hat{w}_i = -\hat{w}_i\left(\frac{\theta_i}{\theta_k} + 1\right). \tag{8}$$

The McFadden shadow elasticity can be written from (7) and (8) as

$$\phi_{ik} \equiv \left.\frac{d\ln(a_i/a_k)}{d\ln(w_k/w_i)}\right|_{dC=0} = 0.5\frac{(\theta_i\mu_{ik} + \theta_k\mu_{ki})}{(\theta_i + \theta_k)}. \tag{9}$$

The symmetric McFadden elasticity is half of a weighted average of the two relevant Morishima elasticities. In computations, ϕ_{ik} will always be positive. If the own price elasticities are more than twice the cross price elasticities in absolute value, ϕ_{ik} will be larger than either of the cross price elasticities ε_{ik} and ε_{ki}.

4. AN ALTERNATIVE MEASURE OF BILATERAL SUBSTITUTION

The ideal would be to report optimal adjustment in relative inputs a_i/a_k when the relative input price w_k/w_i changes, with all inputs flexible and cost minimization. When only w_k and w_i change, the change in a_i can be written

$$\hat{a}_i = \varepsilon_{ik}\hat{w}_k + \varepsilon_{ii}\hat{w}_i. \tag{10}$$

For any particular change in w_k/w_i,

$$\frac{\hat{a}_i}{(\hat{w}_k - \hat{w}_i)} = \frac{(\varepsilon_{ik}\hat{w}_k + \varepsilon_{ii}\hat{w}_i)}{(\hat{w}_k - \hat{w}_i)}. \tag{11}$$

The Hicks-Allen elasticity can be written from (1) and (11) as

$$\begin{aligned} \alpha_{ik} &= \frac{(\hat{a}_i - \hat{a}_k)}{(\hat{w}_k - \hat{w}_i)}, \\ &= \frac{[(\varepsilon_{ik} - \varepsilon_{kk})\hat{w}_k + (\varepsilon_{ii} - \varepsilon_{ki})\hat{w}_i]}{(\hat{w}_k - \hat{w}_i)}, \\ &= \frac{(\mu_{ik}\hat{w}_k - \mu_{ki}\hat{w}_i)}{(\hat{w}_k - \hat{w}_i)}. \end{aligned} \tag{12}$$

The Hicks-Allen elasticity is thus a weighted difference of the two relevant Morishima elasticities.

When $\hat{w}_i = 0$ and $\hat{w}_k \neq 0$, $\alpha_{ik} = \mu_{ik}$. For any given $\hat{w}_i$, l'Hospital's rule implies

$$\lim_{\hat{w}_k \to \hat{w}_i} \alpha_{ik} = \mu_{ik}. \tag{13}$$

The Morishima elasticity thus reflects the ideal Hicks-Allen elasticity in the limit. Generally, however, α_{ik} depends on the actual percentage factor price changes, and would vary along the same isoquant.

An alternative simplifying assumption leads to a bilateral elasticity which is relatively close in value to the underlying ε_{ik} and ε_{ki}. Assume that inputs a_i and a_k are functions only of the difference between their relative factor price changes, $\hat{w}_k - \hat{w}_i$, which amounts to assuming there are only two inputs.

Disregarding other inputs, the bilateral elasticity of substitution is a function of Morishima elasticities:

$$\begin{aligned} \beta_{ik} &= \frac{(\hat{a}_i - \hat{a}_k)}{(\hat{w}_k - \hat{w}_i)} = \left[\frac{(\hat{w}_k - \hat{w}_i)}{\hat{a}_i}\right]^{-1} - \left[\frac{(\hat{w}_k - \hat{w}_i)}{\hat{a}_k}\right]^{-1}, \\ &= (\varepsilon_{ik}{}^{-1} - \varepsilon_{ii}{}^{-1})^{-1} - (\varepsilon_{kk}{}^{-1} - \varepsilon_{ki}{}^{-1})^{-1}, \\ &= \left[\frac{(\varepsilon_{ii} - \varepsilon_{ik})}{\varepsilon_{ii}\varepsilon_{ik}}\right]^{-1} - \left[\frac{(\varepsilon_{ki} - \varepsilon_{kk})}{\varepsilon_{kk}\varepsilon_{ki}}\right]^{-1}, \\ &= \left(\frac{-\mu_{ik}}{\varepsilon_{ik}\varepsilon_{ii}}\right)^{-1} - \left(\frac{\mu_{ki}}{\varepsilon_{kk}\varepsilon_{ki}}\right)^{-1}, \\ &= -\left(\frac{\varepsilon_{ii}\varepsilon_{ik}}{\mu_{ik}} + \frac{\varepsilon_{ki}\varepsilon_{kk}}{\mu_{ki}}\right), \\ &= -\frac{(\varepsilon_{ii}\varepsilon_{ik}\mu_{ki} + \varepsilon_{ki}\varepsilon_{kk}\mu_{ik})}{\mu_{ik}\mu_{ki}}. \end{aligned} \tag{14}$$

Since $\varepsilon_{jj} < 0$ for every j, β_{ik} will be positive if factors i and k are substitutes ($\varepsilon_{ki}, \varepsilon_{ik} > 0$) and negative if they are complements ($\varepsilon_{ik}, \varepsilon_{ki} < 0$). The bilateral elasticity β_{ik} is not symmetric, and typically will be closer in value to ε_{ik} and ε_{ki} than will μ_{ik}. With Cobb-Douglas or CES production functions, β_{ik} will be constant. Between two particular inputs, β_{ik} will be constant if the production function is separably CES in those inputs.

A model with three inputs is the simplest situation where this bilateral elasticity is relevant. Imagine it is a smooth bowl isoquant, just supported by an isocost plane at a tangency which determines the cost minimizing inputs a_i. Endpoints of the supporting isocost plane on each of the axes are C/w_i. The input price ratio w_2/w_1 is (the negative of the) slope of the line from the endpoints on the 1 and 2 axes. A change in w_2/w_1 means a change in the slope of this edge of the supporting isocost plane.

In a Morishima adjustment, only w_2 changes, with both w_1 and w_3 held constant. This is pictured by a shift of the isocost endpoint C/w_2. The Morishima elasticity μ_{12} measures the constrained change in inputs 1 and 2 in the move to the new cost minimization. Cost adjusts and the isocost plane shifts.

The McFadden elasticity holds cost constant, allowing for a change in w_2/w_1. Endpoint C/w_3 of the isocost plane is thus fixed. A McFadden adjustment overstates the adjustment which would occur if cost were free to adjust.

The bilateral elasticity β_{ik} measures optimal adjustment in the input ratio a_1/a_2 due to a change in w_2/w_1 with the input of factor 3 fixed at the original cost minimizing a_3. Cost fully adjusts, subject to the fixed a_3. Substitution occurs along a slice of the isoquant in the a_3 plane.

5. A COMPARISON OF ELASTICITIES

The following example comes from [9], where substitution between production workers P, nonproduction workers N, and capital K, is estimated under a translog production structure in U.S. manufacturing across states according to two-digit SIC classifications. Factor shares are $\theta_P = .503$, $\theta_N = .251$, and $\theta_K = .246$. Estimated cross price elasticities are

$$\begin{bmatrix} \varepsilon_{PP} & \varepsilon_{PN} & \varepsilon_{PK} \\ \varepsilon_{NP} & \varepsilon_{NN} & \varepsilon_{NK} \\ \varepsilon_{KP} & \varepsilon_{KN} & \varepsilon_{KK} \end{bmatrix} = \begin{bmatrix} -.334 & .092 & .242 \\ .184 & -.658 & .474 \\ .494 & .237 & -.731 \end{bmatrix}. \tag{15}$$

From (15), the corresponding Morishima elasticities are

$$\begin{bmatrix} - & \mu_{PN} & \mu_{PK} \\ \mu_{NP} & - & \mu_{NK} \\ \mu_{KP} & \mu_{KN} & - \end{bmatrix} = \begin{bmatrix} - & .750 & .973 \\ .518 & - & 1.21 \\ .828 & .895 & - \end{bmatrix}. \tag{16}$$

Note that the Morishima elasticities are much larger than the cross price elasticities, and would have been positive even if cross price elasticities had been negative. Although all cross price elasticities in (15) are inelastic, elasticity is suggested between nonproduction workers and capital by the Morishima elasticities in (16).

Corresponding McFadden elasticities are

$$\begin{bmatrix} - & \phi_{PN} & \phi_{PK} \\ & - & \phi_{NK} \\ & & - \end{bmatrix} = \begin{bmatrix} - & .337 & .463 \\ & - & .525 \\ & & - \end{bmatrix}. \tag{17}$$

The McFadden elasticities in (17) are symmetric and larger than the cross price elasticities in (15). Reporting either Morishima or McFadden elasticities would be misleading.

The bilateral elasticities of substitution from (14) are

$$\begin{bmatrix} - & \beta_{PN} & \beta_{PK} \\ \beta_{NP} & - & \beta_{NK} \\ \beta_{KP} & \beta_{KN} & - \end{bmatrix} = \begin{bmatrix} - & .248 & .519 \\ .275 & - & .451 \\ .534 & .340 & - \end{bmatrix}. \tag{18}$$

6. CONCLUSION

The bilateral elasticity derived in this note is a viable alternative to Morishima and McFadden elasticities for reporting summary substitution elasticities when there are many inputs. This bilateral elasticity is generally closer in value to the underlying cross price elasticities than the Morishima elasticity, is not symmetric like the McFadden elasticity, and reflects underlying complementarity when it occurs.

REFERENCES

1. J.R. Hicks, *Theory of Wages*, MacMillan, London, (1932).
2. R.G.D. Allen, *Mathematical Analysis for Economics*, MacMillan, London, (1938).
3. M. Morishima, A few suggestions on the theory of elasticity, (in Japanese), *Keizai Hyoron (Economic Review)* **16**, 144–150 (1967).
4. D. McFadden, Constant elasticities of substitution production functions, *Review of Economic Studies* **31**, 73–83 (March 1963).
5. R.G. Chambers, *Applied Production Analysis: A Dual Approach*, pp. 93–100, Cambridge University Press, Cambridge, (1988).
6. C. Blackorby and R.R. Russell, Will the real elasticity of substitution please stand up (A comparison of the Allen/Uzawa and Morishima elasticities), *American Economic Review* **79**, 882–888 (September 1989).
7. Y. Mundlack, Elasticities of substitution and the theory of derived demand, *Review of Economic Studies* **35**, 225–236 (1968).
8. C. Blackorby and R.R. Russell, The partial elasticity of substitution, Discussion Paper No. 75-1, Economics, University of California, San Diego, CA, (1975).
9. R.B. Freeman and J.L. Medoff, Substitution between production labor and other inputs in unionized and nonunionized manufacturing, *The Review of Economics and Statistics* **64**, 220–233 (1982).