MAX-PLANCK-GESELLSCHAFT

Statistical Mechanics where Newton's Third Law is Broken

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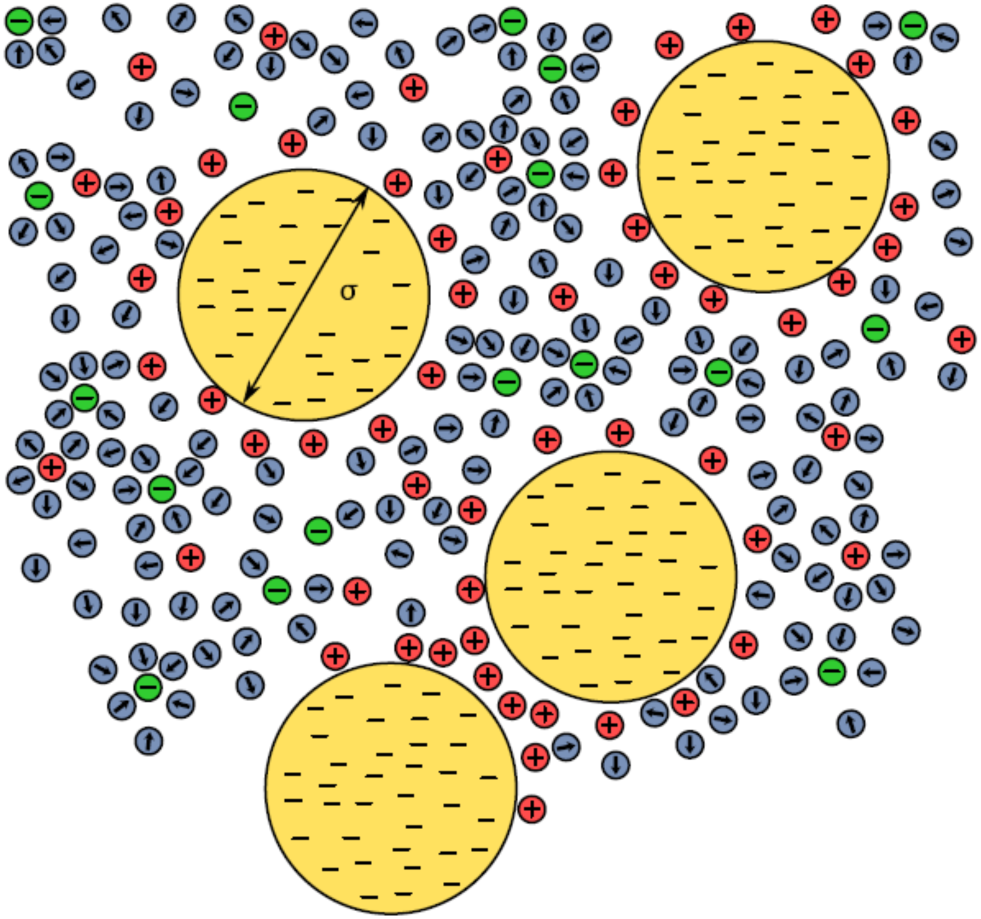
Effective interactions

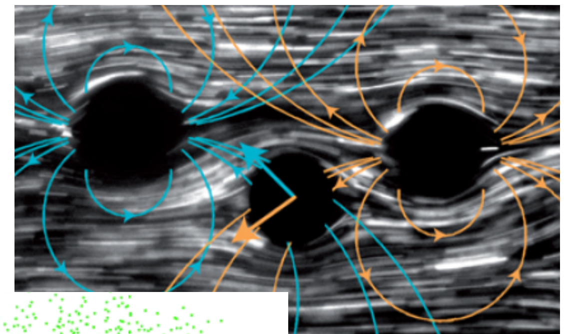


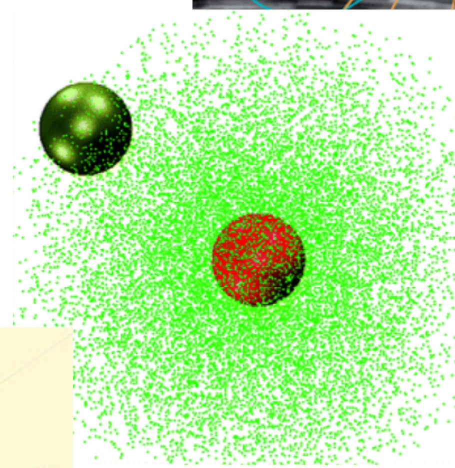
Newton's third law holds for the **effective** interactions in *equilibrium.*

However, real life is nonequilibrium

Examples of nonreciprocal effective forces

- Interactions between colloids under solvent or depletant flow
- Wake-mediated interactions in a flowing plasma





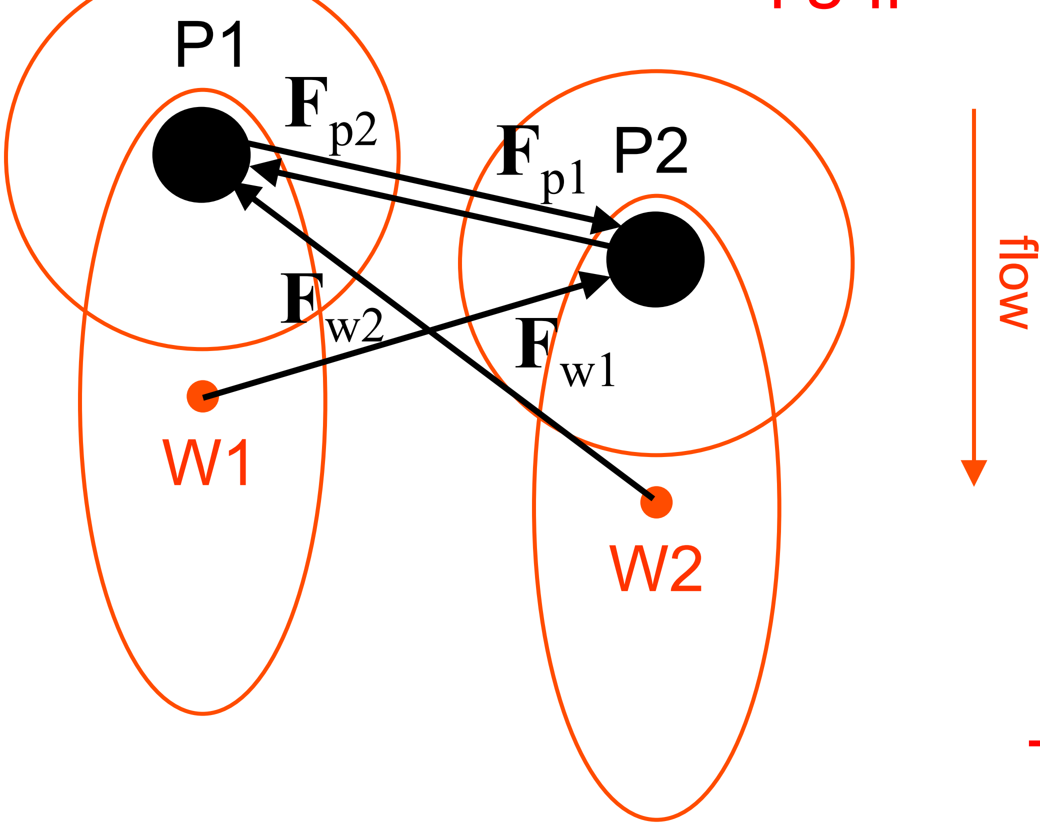
- Diffusiophoretic forces between colloids
- Neutral shadow forces

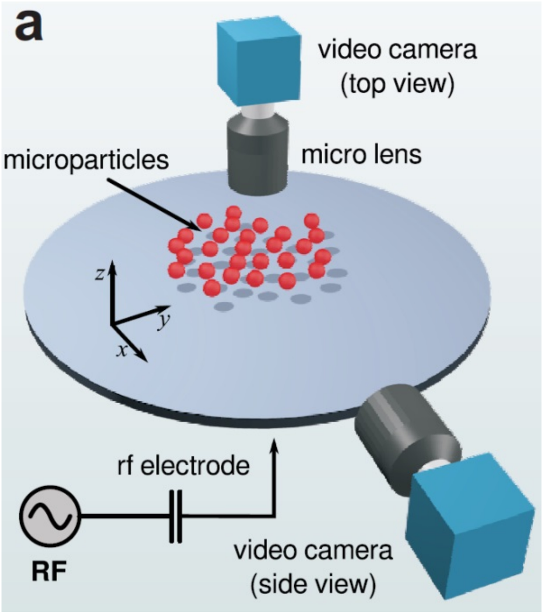
- “Social” forces

Non-reciprocity of the wake-mediated interactions

isotropic plasma

flowing plasma



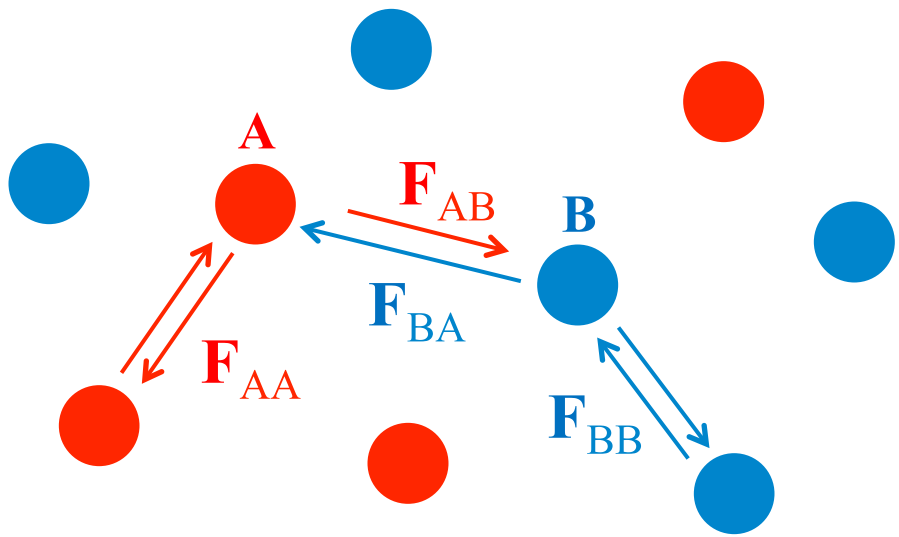


$$\mathbf{F}_{12}+\mathbf{F}_{21} \neq \mathbf{0}$$

The action-reaction symmetry is broken

V. Schweigert et al., Phys. Rev. E 54, 4155 (1996)

Generic form for nonreciprocal pair interactions



$$\mathbf{F}_{ij} = -\frac{\partial \varphi(r_{ij})}{\partial \mathbf{r}_j} \times \begin{cases} 1-\Delta & \text{for } ij \in AB \\ 1+\Delta & \text{for } ij \in BA \\ 1 & \text{for } ij \in AA \text{ or } BB \end{cases}$$

Renormalization of masses and interactions

$$\tilde{m}_i = m_i \times \begin{cases} (1+\Delta)^{-1} & \text{for } i \in A \\ (1-\Delta)^{-1} & \text{for } i \in B \end{cases}$$

$$\tilde{\varphi}(r_{ij}) = \varphi(r_{ij}) \times \begin{cases} (1+\Delta)^{-1} & \text{for } ij \in AA \\ (1-\Delta)^{-1} & \text{for } ij \in BB \\ 1 & \text{for } ij \in AB \text{ or } BA \end{cases}$$

$$\Downarrow$$

$$\tilde{m}_i \dot{\mathbf{v}}_i = -\frac{\partial}{\partial \mathbf{r}_i} \sum_j^N \tilde{\varphi}(r_{ij})$$

Interactions become reciprocal!

Pseudo-Hamiltonian systems

$$\tilde{H} = \sum_{i}^{N} \frac{1}{2} \tilde{m}_i v_i^2 + \sum_{i<j}^{N} \tilde{\varphi}(r_{ij})$$

Detailed equilibrium is possible, with the energy equipartition:

$$\tfrac{1}{2}\tilde{m}_A \langle v_A^2 \rangle \; = \; \tfrac{1}{2}\tilde{m}_B \langle v_B^2 \rangle$$

$$\Downarrow$$

$$\frac{T_A}{T_B} = \frac{1+\Delta}{1-\Delta}$$

Pseudo-equilibrium

A system of particles with nonreciprocal interactions, being essentially nonequilibrium, can nevertheless be exactly described in terms of equilibrium statistical mechanics.

$\Downarrow$

A mixture of two fluids can be in detailed *dynamic* equilibrium, when each fluid has its own temperature.

General case of non-reciprocity

Isotropic pair interactions can always be presented as follows:

$$\mathbf{F}_{AB,BA}(r) = \mp\mathbf{F}_{\mathrm{r}}(r) + \mathbf{F}_{\mathrm{n}}(r)$$

$$\mathbf{F}_{\mathrm{r,n}}(r) = -\frac{d\varphi_{\mathrm{r,n}}(r)}{dr}\frac{\mathbf{r}}{r}$$

Constant non-reciprocity is recovered when $F_{\mathrm{n}}(r)/F_{\mathrm{r}}(r) \to \Delta$

Asymptotic temperature ratio (no damping)

All nonreciprocal systems have a universal asymptotic behavior:

$$t \to \infty : \quad T_A(t) = \tau T_B(t) = ct^{2/3} + \text{const}$$

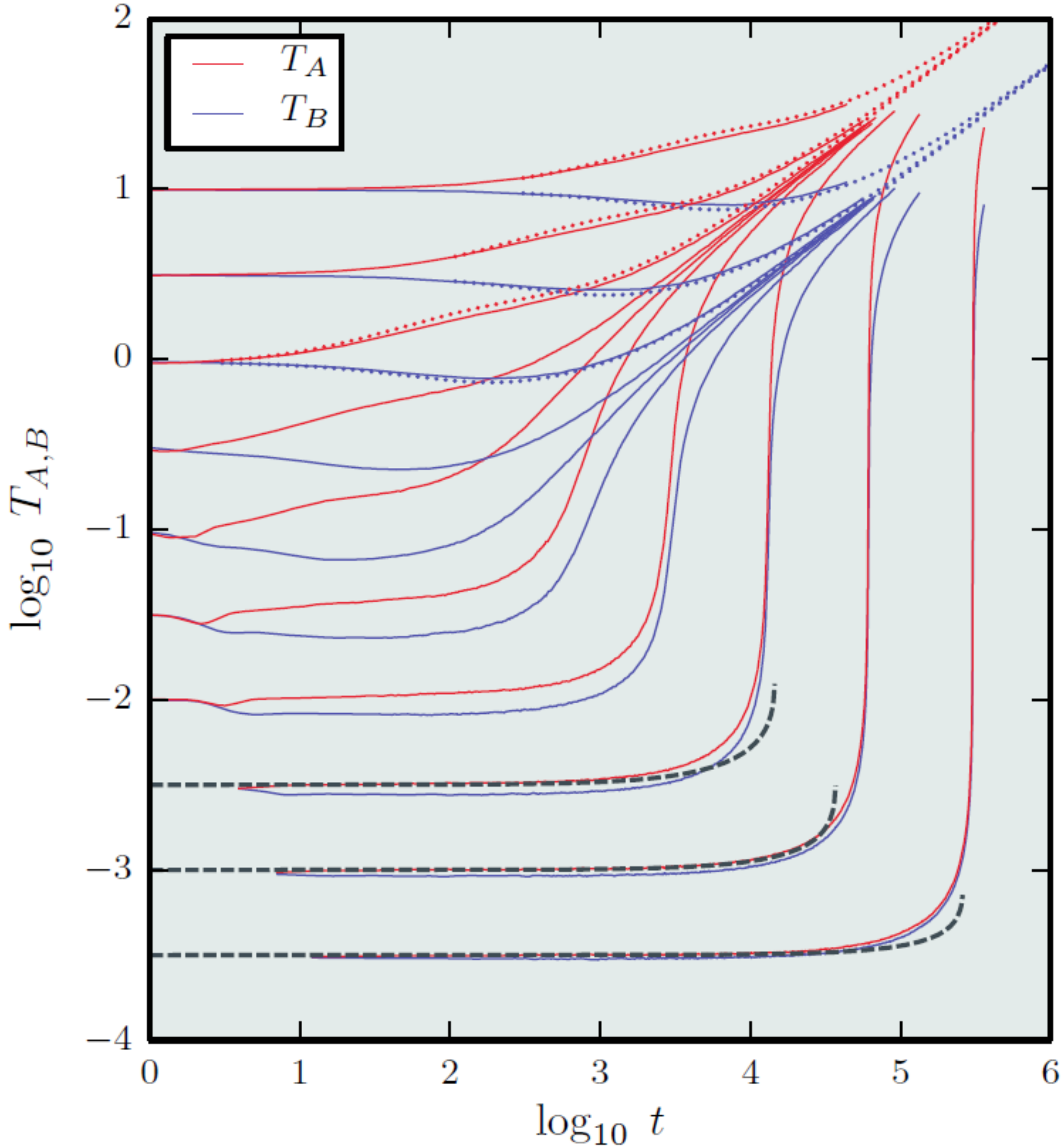
The constants are determined by the effective non-reciprocity Δ_{eff} and disparity ε:

$$\tau = \sqrt{\frac{(1+\Delta_{\text{eff}})^2 + \epsilon}{(1-\Delta_{\text{eff}})^2 + \epsilon}} \qquad c = \text{const}\ \epsilon^{2/3}$$

Constant non-reciprocity: $F_n(r)/F_r(r) \to \Delta,\ \Delta_{\text{eff}} \to \Delta,\ \varepsilon \to 0$

$$\tau \to \frac{1+\Delta}{1-\Delta} \qquad c \to 0$$

Numerical example



Elastic hard spheres (Hertzian interactions):

$$\Delta_{\mathrm{eff}} = 0.57,\ \varepsilon = 0.082$$

$$\Downarrow$$

$$T_A / T_B \,|_{t\to\infty} \approx 3.1$$

Weak finite damping leads to a steady state, while T_A/T_B does not (practically) change.

Role of damping

Dynamics of individual particles in the presence of damping is governed by:

$$m\dot{\mathbf{v}}_i + m\nu\mathbf{v}_i = \mathbf{F}_i(\{\mathbf{r}\}) + \mathbf{L}_i(t) \qquad \langle \mathbf{L}_i(t+\tau)\mathbf{L}_j(t)\rangle = 2m\nu k_{\mathrm{B}}T\delta_{ij}\delta(\tau)$$

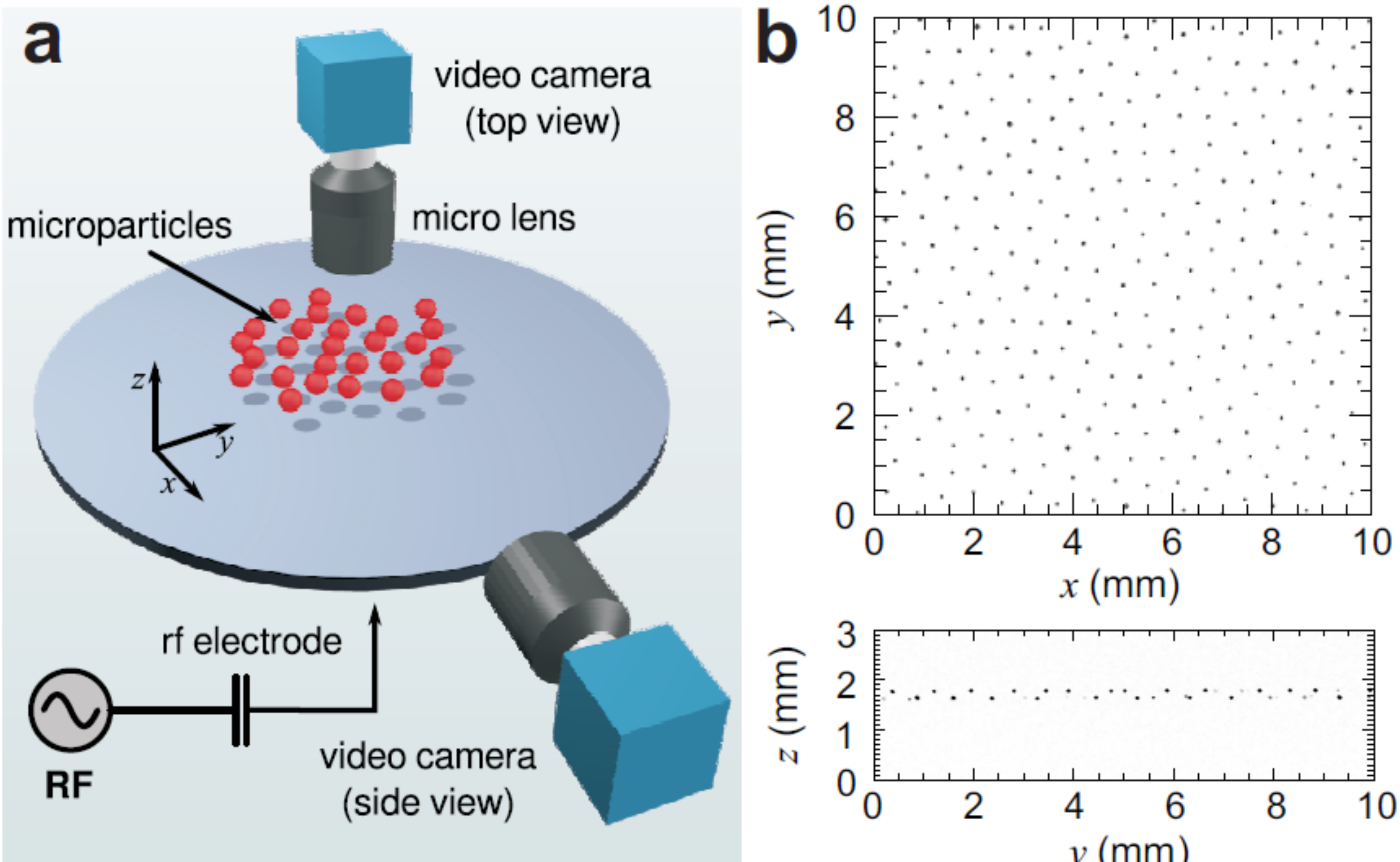
A weak-damping regime is defined as $\nu \ll \nu_{\mathrm{coll}}$: the relevant collision frequency ν_{coll} should be much smaller than the damping rate ν.

$\Downarrow$

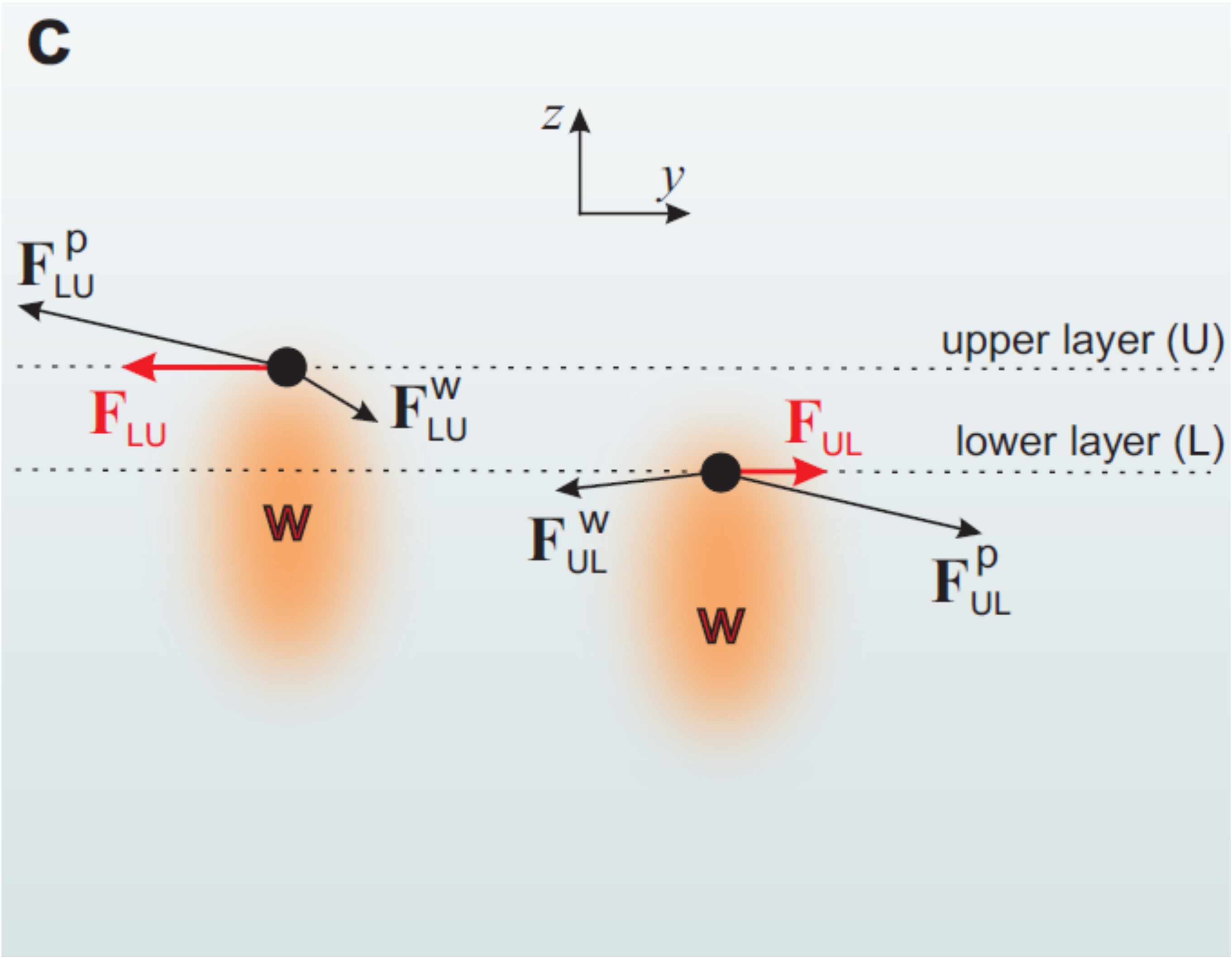
$$T_{A,B} = \frac{T}{1 \mp \Delta} + O\left(\frac{\nu}{\nu_{\mathrm{coll}}}\right)$$

The temperature ratio is not affected!

Experiment with 2D binary mixture



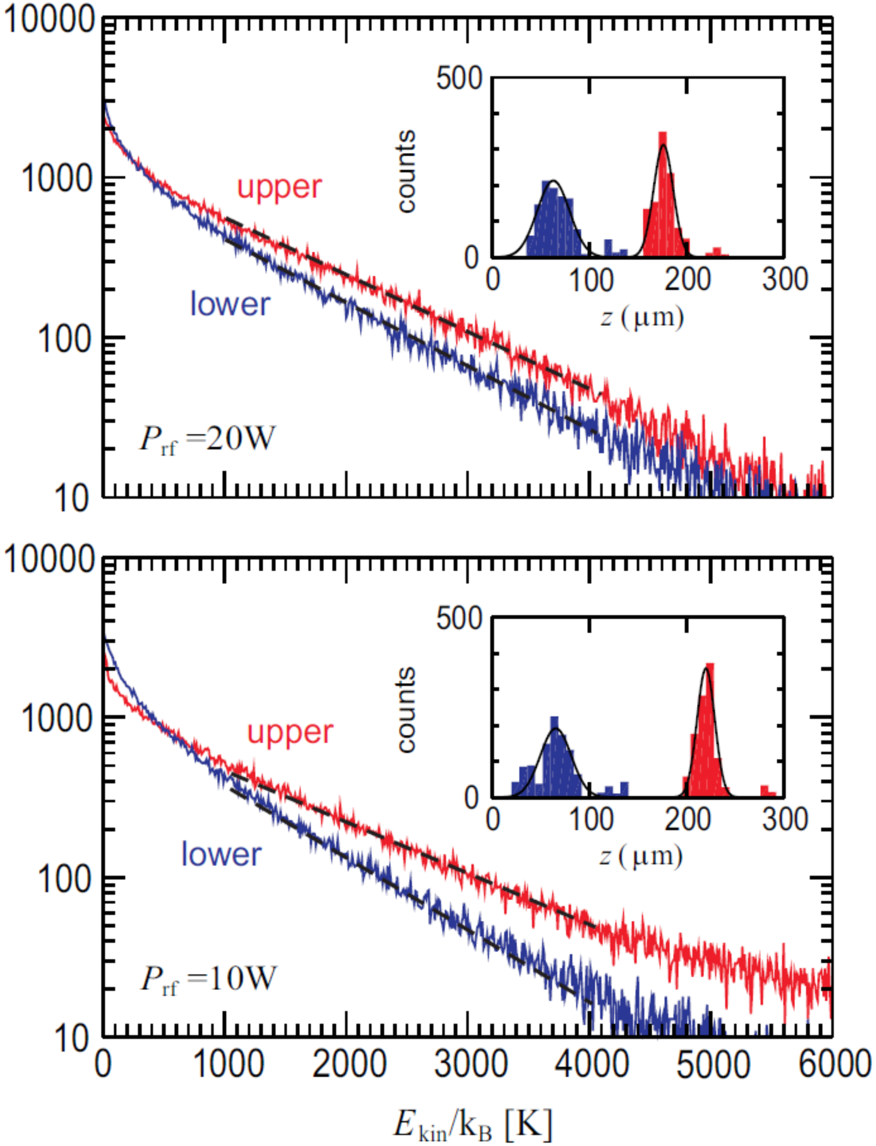
Nonreciprocal forces



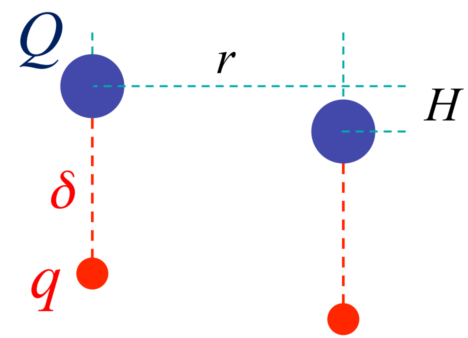
$\mathbf{F}_{LU}=(1+\Delta)\mathbf{F}_0$

$\mathbf{F}_{UL}=(1-\Delta)\mathbf{F}_0$

Kinetic energy distribution



Point-wake model:



Non-reciprocity parameter:

$$\Delta(r) \propto (q/Q)H\delta/r^2$$

$$\frac{T_U}{T_L} = \frac{1+\Delta}{1-\Delta} \simeq 1 + 2\Delta$$

Conclusions

- The action-reaction symmetry breaking makes systems of interacting particles essentially nonequilibrium.

  However, for a constant non-reciprocity ($\Delta = \text{const}$) such systems can be exactly mapped on Hamiltonian systems and, hence, be described with the equilibrium statistical mechanics.

- A mixture of particles in this case reaches detailed dynamic equilibrium, when each species has its own temperature.

- For a general case, $\Delta(r)$, the system is no longer in equilibrium. However, the temperature ratio tends to a universal constant value.

- A weak damping does not (practically) affect the results.

- The principal theoretical predictions have been verified in experiments with 2D binary complex plasmas.

Outlook

- **Brownian dynamics for nonreciprocal interactions**

In a system with the fully damped dynamics, the kinetic temperatures of all species are the same. However, one can show that the dynamics with nonreciprocal interactions is equivalent to the reciprocal dynamics with the renormalized thermostat temperatures:

$$\tilde{T}_{A,B} = \frac{T}{1 \pm \Delta}$$

This leads to a rich variety of self-organization phenomena, such as dynamical clustering (active quasi-particles), pattern formation (quasi-nematic phases), phase separation, etc.

Outlook

- **“Nearly Hamiltonian” systems**

In a general case of non-reciprocity with $\Delta(r)$, a system reaches a (nonequilibrium) steady state. The temperature ratio is characterized by the non-reciprocity Δ_{eff} and disparity ε. When $\varepsilon \ll 1$, the ratio tends to

$$\tau \simeq \frac{1 + \Delta_{\mathrm{eff}}}{1 - \Delta_{\mathrm{eff}}}$$

Question: How “far” are such systems from a pseudo-Hamiltonian system with $\Delta = \Delta_{\mathrm{eff}}$? What are the signatures of deviation from the pseudo-equilibrium ?

General case of non-reciprocity

Isotropic pair interactions can always be presented as follows:

$$\mathbf{F}_{AB,BA}(r) = \mp\mathbf{F}_{\rm r}(r) + \mathbf{F}_{\rm n}(r)$$

$$\mathbf{F}_{\rm r,n}(r) = -\frac{d\varphi_{\rm r,n}(r)}{dr}\frac{\mathbf{r}}{r}$$

In the center-of-mass ($\mathbf{R}$) and relative ($\mathbf{r}$) coordinates we have:

$$M\ddot{\mathbf{R}} = 2\mathbf{F}_{\rm n}(r)$$

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{\rm r}(r) + \frac{m_B - m_A}{m_A + m_B}\mathbf{F}_{\rm n}(r)$$

The kinetic energy of the CoM motion is not conserved

Asymptotic temperature ratio

All non-reciprocal systems have a universal asymptotic behavior:

$$t \to \infty : \quad T_A(t) = \tau T_B(t) = c t^{2/3} + \mathrm{const}$$

The constant τ is determined by the effective non-reciprocity Δ_{eff} and disparity ε:

$$\tau = \sqrt{\frac{(1+\Delta_{\mathrm{eff}})^2 + \epsilon}{(1-\Delta_{\mathrm{eff}})^2 + \epsilon}} \qquad c = \mathrm{const}\ \epsilon^{2/3}$$

$$\Delta_{\mathrm{eff}} = \frac{I_{\mathrm{nn}}}{I_{\mathrm{rn}}} \qquad I_{\alpha\beta} = \int_0^\infty d\rho\, f_\alpha f_\beta$$

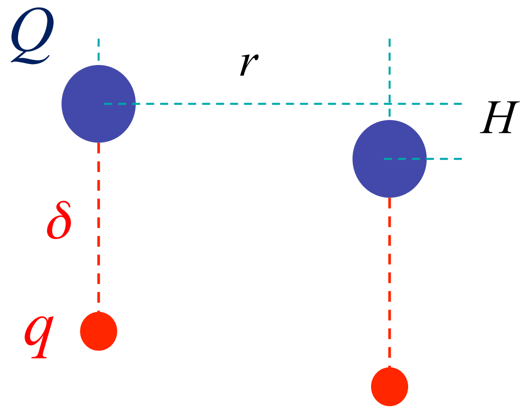
$$\epsilon = \frac{I_{\mathrm{rr}} I_{\mathrm{nn}}}{I_{\mathrm{rn}}^2} \qquad f_\alpha(\rho) = \int_\rho^\infty dr \frac{F_\alpha(r)}{\sqrt{r^2-\rho^2}}$$

$F_n(r)/F_r(r) \to \Delta$: $\quad \Delta_{\mathrm{eff}} \to \Delta, \quad \varepsilon \to 0$

$$\tau \to \frac{1+\Delta}{1-\Delta}$$

$$c \to 0$$

Point-wake model



$$\varphi_{\mathrm{r}}(r) \;=\; \varphi_Q(r,H) - \frac{1}{2}\left[\varphi_q(r,H) + \varphi_q(r,-H)\right],$$

$$\varphi_{\mathrm{n}}(r) \;=\; \frac{1}{2}\left[\varphi_q(r,H) - \varphi_q(r,-H)\right].$$

Effect of weak damping: General non-reciprocity

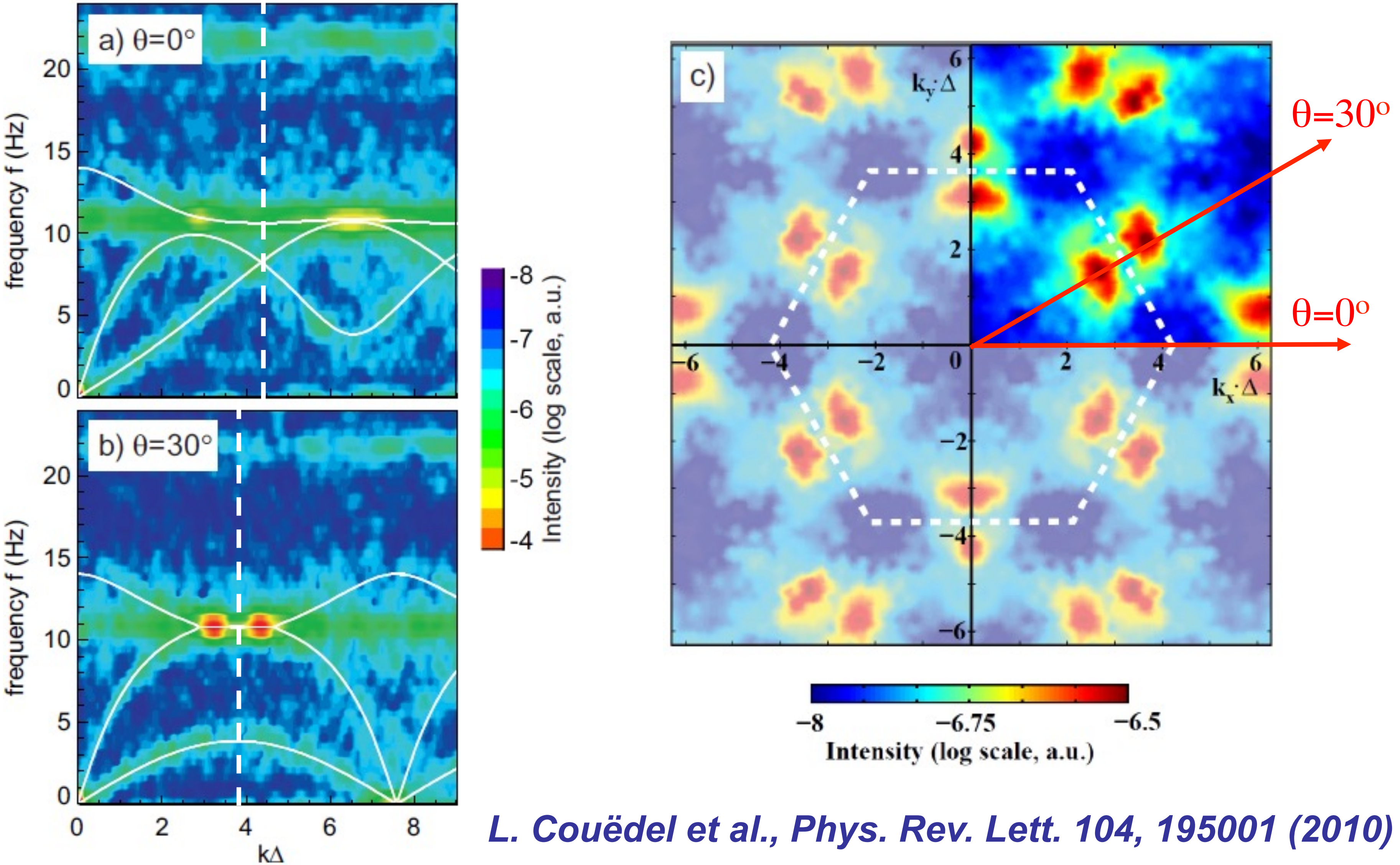
The temperature ratio in the presence of damping, $T_A/T_B = \tau_\nu$, is given by:

$$\tilde{\nu}[(1-\Delta_{\rm eff})^2+\epsilon]\tau_\nu^2-(\tilde{\nu}-1)(1-\Delta_{\rm eff}^2+\epsilon)\tau_\nu = (1+\Delta_{\rm eff})^2+\epsilon$$

$$\tilde{\nu} = \nu_A/\nu_B$$

When $\nu_A=\nu_B$, the temperature ratio is equal to the undamped value τ.

Theory vs. Experiment: Mode-coupling instability



L. Couëdel et al., Phys. Rev. Lett. 104, 195001 (2010)