The q-numerical range and the real q-numerical range of the shifts

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Abstract

We show that the q-numerical range and the real q-numerical range of the unilateral shift and bilateral shift and their adjoints are the closed unit disc when $|q| < 1$. We also consider some weighted shifts.

The *classical numerical range* of an operator $A \in B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ is the set $W(A) = \{(Ax, x) : x \in S\}$ in $\mathbb{C}$ where $S = \{x \in \mathcal{H}, (x, x) = 1\}$. It is always convex [2], [3]. If $\mathcal{H}$ is separable, the unilateral shift U is defined by $Ue_i = e_{i+1}$, where $\{e_i\}_{i=1}^{\infty}$ is an orthonormal basis for $\mathcal{H}$. The bilateral shift V is defined similarly [3]. It is well-known that $W(U)$ and $W(V)$ are the open unit disc [3]. A generalization of the classical numerical range is the *q-numerical range*:

$$W_q(A) = \{(Ax, y) : (x, y) = q, x, y \in S\},$$

where $|q| \leq 1$. It is known that $W_q(A)$ is always convex [10] and $W_1(A) = W(A) = -W_{-1}(A)$. The *$q$-numerical radius* of A is denoted by $w_q(A) = \sup\{|z| : z \in W_q(A)\}$. For finite dimensional shift N_n, $W_q(N_n)$ is a closed disc centered at the origin, but $w_q(A)$ can be complicated when $|q| < 1$. When $n = 2$, $w_q(N_2) = \frac{1}{2}(1 + \sqrt{1 - q^2})$. When $n = 3$

$$w_q(N_3) = \frac{1}{8}\sqrt{27 + 18q - 13q^2 + (9 + 7q)\sqrt{(1 - q)(9 + 7q)}}, \quad \text{if} \quad \frac{1}{2} \leq q \leq 1,$$

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and $w_q(N_3) = 1$ if $0 \le q \le \frac{1}{2}$ [4]. In particular $w_1(N_3) = \frac{1}{\sqrt{2}}$. Beside that not much is known about $w_q(N_n)$, $|q| < 1$.

Let $q \in [-1, 1]$. We also consider a different set

$$R_q(A) = \{(Ax, y) : \mathrm{Re}\,(x, y) = q, x, y \in S\},$$

and we call it the *real q-numerical range* of A. The radius $r_q(A)$ is defined similarly. Clearly $R_{-q}(A) = -R_q(A)$, $W_q(A) \subset R_q(A)$ providing q is real and $R_{\pm 1}(A) = W_{\pm 1}(A) = \pm W(A)$. Recently it is known [6] that $R_q(A)$ is always convex.

Lemma 1 *Let $A \in B(\mathcal{H})$. Then*

1. *$W_{qz}(A) = zW_q(A)$ for any $|z| = 1$. Hence if $W_q(A)$ is a disc, then $W_q(A) = W_p(A)$ for all $|p| = |q|$.*

2. *$W_q(A)$ is invariant under unitary equivalence, i.e., $W_q(A) = W_q(U^*AU)$ if U is a unitary operator.*

3. *$R_q(A) = \cup_{Re\, p=q} W_p(A)$.*

The following result answers a question mentioned in [4].

Theorem 2 *The q-numerical range and the real q-numerical range of the unilateral shift and the bilateral shift and their adjoints are the closed unit disc $D(0, 1)$ if $|q| < 1$.*

Proof: Let U be the unilateral shift. Notice that [5] $W_q(U)$ is a disc centered at the origin. Due to Lemma 1 we can assume that $q \in [0, 1)$. It is clear that $w_q(A) \le \|A\|$ for $A \in B(\mathcal{H})$ and $q \in [0, 1]$. If $0 \le q < 1$, let

$$\begin{aligned} x &= \sqrt{1 - q^2}\,(1, q, q^2, q^3, \ldots), \\ y &= Ux = \sqrt{1 - q^2}\,(0, 1, q, q^2, q^3, \ldots). \end{aligned}$$

Then $x, y \in S$, $(x, y) = q$ and $(Ux, y) = (y, y) = 1$. So $w_q(U) = 1$ and $W_q(U)$ is closed if $0 \le q < 1$. Now $W_q(A^*)$ is the complex conjugate of $W_q(A)$ if $A \in B(\mathcal{H})$, $q \in [0, 1]$. So $W_q(U^*) = D(0, 1)$.

Now $W_q(U) \subset R_q(U) \subset D(0, 1)$. Thus $R_q(U) = D(0, 1)$. For the bilateral shift V, one has the conclusion by observing $W_q(U) \subset W_q(V) \subset D(0, 1)$ and $R_q(U) \subset R_q(V) \subset D(0, 1)$. The same conclusion is true for the adjoint of V. $\square$

We can extend the above result to some special weighted shifts. A unilateral weighted shift S is an operator defined by $Se_n = s_n e_{n+1}$, $n > 0$, and bilateral weighted shift is definely analogously. Notice that S is unitarily equivalent to a shift T with weights t_n where $|t_n| = |s_n|$, for all n. Due to Lemma 4 we can assume that $s_n \ge 0$.

Theorem 3 *1. Let S be a unilateral weighted shift with weights $\{s_n\}$. If $s_n \le K$ for all positive integers n, and $\lim_{n\to\infty} s_n = K > 0$, then*

(a) [9] $W_1(S)$ and $W_1(S^*)$ are the open disc centered at the origin with radius K,

(b) when $|q| < 1$, $W_q(S)$ and $W_q(S^*)$ are a disc centered at the origin with radius K. In addition,

i. $W_0(S)$ and $W_0(S^*)$ are closed if and only if $s_n = K$ for some n.

ii. when $0 < |q| < 1$, then $W_q(S)$ and $W_q(S^*)$ are closed if and only if there are infinitely many n such that $s_n = K$.

2. The statements are true for the bilateral weighted shift with weights $\{s_n\}$ such that $s_n \le K$ for all integers n, and $\lim_{n\to\infty} s_n = K > 0$ or $\lim_{n\to\infty} s_{-n} = K > 0$

3. The statements are true for the real q-numerical range of the corresponding weighted shifts where $q \in [-1, 1]$.

Proof: 1 (b) Let $|q| < 1$ be real since $W_{qz}(A) = zW_q(A)$ for any $|z| = 1$ and $A \in B(\mathcal{H})$. Notice that $W_q(S)$ is a disc centered at the origin [5]. Now let $x = \sqrt{1-q^2}(0, \ldots, 0, 1, q, q^2, \ldots)$ where the first n entries are zero and $y = Ux$ where U is the unilateral shift. Then $\|x\| = \|y\| = 1$, $(x, y) = q$ and $(Sx, y) = (1-q^2)\sum_{i=n}^{\infty} s_i q^{2i}$. For any $\epsilon > 0$, there exists N such that $s_n \ge K - \epsilon$ whenever $n > N$. Hence if $n > N$, then $(1-q^2)\sum_{i=n}^{\infty} s_i q^{2i} \ge (K-\epsilon)(1-q^2)\sum_{i=n}^{\infty} q^{2i} = K - \epsilon$. Thus $(Sx, y) \to K$ as $n \to \infty$. Now $K \le w_q(S) \le \|S\| = K$ so that $w_q(S) = K$.

(i) Now $W_0(S)$ is the closed disc $D(0, K)$ if and only if there are orthogonal unit vectors x and y such that $K = (Sx, y)$. When $s_n = K$ for some n, we have $(Se_n, e_{n+1}) = K$. So $W_0(S) = D(0, K)$.

Suppose that $s_n < K$ for all n. If $W_0(S)$ were $D(0, K)$, then $K = (Sx, y) \le \|Sx\|\|y\| \le \|S\| = K$ for some orthogonal unit vectors x and y. So $\|Sx\| = K$. But $\|Sx\|^2 = \sum_{n=1}^{\infty} s_n^2 |x_n|^2 < K^2\|x\|^2 = K^2$. It is absurd.

(ii) Now we assume that $0 < |q| < 1$. If there are infinitey many n such that $s_n = K$, say, $n_1, n_2, \ldots$, let x be the unit vector such that $x_{n_i} = q^{i-1}/\sqrt{1-q^2}$, $i = 1, 2, \ldots$ and zeros otherwise, and y be Ux where U is the unilateral shift. Then $(Sx, y) = K$ and hence $W_q(S) = D(0, K)$.

Suppose there are finitely many n such that $s_n = K$, say, $n_1, \ldots, n_k$. If $W_q(S) = D(0, K)$, then we would have unit vectors x, y such that $(x, y) = q$ and $(Sx, y) = K$. So $K = (Sx, y) \le \|Sx\|\|y\| \le K$. Thus we have $\|Sx\| = K$ and Sx and y scalar multiple to each other, say, $y = \lambda Sx$. Now $K^2 = \|Sx\|^2 = \sum_{i=1}^{\infty} s_i^2 |x_i|^2$. So $x_i = 0$ except $i = n_j$ for some $j = 1, \ldots, k$. Then $q = (x, y) = (x, \lambda Sx) = \lambda K(x, Ux) = 0$, a contradiction.

The above proof works for $R_q(S)$ except the condition $(x, y) = q$ in the proof of (ii) is replaced by $\mathrm{Re}\,(x, y) = q$. One can have the same conclusion by using $R_q(S) = \cup_{\mathrm{Re}\, p=q} W_p(S) = \cup_{q\le|p|, p\in\mathbb{C}} W_p(S) = \cup_{p\in[q,1]} W_p(S) = W_q(S)$ if $0 \le q < 1$, since each $W_p(S)$ is a disc. The treatment for the bilateral case is similar. $\square$

Example 4 *1. Let S be the weighted shift such that $Se_n = \frac{n-1}{n} e_{n+1}$, $n \ge 1$. Then $W_q(S)$, $W_q(S^*)$, $R_q(S)$ and $R_q(S^*)$ are the open unit disc for all $|q| \le 1$.*

2. Let T be the weighted shift such that $Te_1 = e_2$, $Te_n = \frac{n-1}{n} e_{n+1}$, $n \ge 2$. Then (i) $W_q(T)$, $W_q(T^*)$, $R_q(T)$ and $R_q(T^*)$ are the open unit disc for all $0 < |q| \le 1$ and (ii) $W_0(T)$ and $W_0(T^*)$ are the closed unit disc.

The following result implies that $W_q(S)$ and $R_q(S)$, $q \in [-1, 1]$ are very much the same for weighted shifts S.

Theorem 5 *Let A be an operator on a Hilbert space such that $W_q(A)$ is a disc centered at the origin for each $|q| \le 1$. Then*

1. $R_q(A)^- = W_q(A)^-$, i.e., $R_q(A)\backslash W_q(A)$ is either a circle or an empty set.

2. $R_{q_2}(A) \subset R_{q_1}(A)$ whenever $0 \le q_1 < q_2 \le 1$.

Proof: (1) It is sufficient to consider $q \in [0, 1]$. Notice that $W_q(A) \subset R_q(A) = \cup_{p\in[q,1]} W_p(A)$. By the decreasing property of the diameter function of A [4], i.e, $d_{q_2}(A) \le d_{q_1}(A)$ if $0 \le q_1 \le q_2 \le 1$ where $d_q(A) = \max\{|z_1 - z_2| : z_1, z_2 \in W_q(A)\}$, and in our case $w_q(A) = \frac{1}{2} d_q(A)$, we have $\cup_{p\in[q,1]} W_p(A) \subset W_q(A)^-$.

(2) $R_{q_2}(A) = \cup_{p\in[q_2,1]} W_p(A) \subset \cup_{p\in[q_1,1]} W_p(A) = R_{q_1}(A)$, $0 \le q_1 \le q_2 \le 1$. $\square$

Question: Is $W_q(S) = R_q(S)$, $q \in [-1, 1]$ always true for weighted shifts S? Or equivlently, is it true that $W_{q_2}(S) \subset W_{q_1}(S)$ whenever $0 \le q_1 < q_2 \le 1$?

Remark 6 When S is a finite dimensional weighted shift, i.e., the matrix representation of S is

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 \\ s_1 & 0 & \dots & 0 \\ & s_2 & \dots & 0 \\ & & & \\ & & s_{n-1} & 0 \end{pmatrix},$$

we have $R_q(S) = \cup_{q\le|p|} W_q(S) = \cup_{p\in[q,1]} W_p(S)$, $q \in [0, 1]$ since $W_q(S)$ is a closed disc centered at the origin. By the decreasing property of the diameter function of A, we have $R_q(S) = W_q(S)$ when $q \in [-1, 1]$.

Remark 7 Generally the q-numerical ranges or the real q-numerical ranges of the weighted shifts are not easy to handle. Shields [1] has considered the weighted shift S such that $Se_1 = (1+h)e_2$, $h \geq 0$ $Se_n = e_{n+1}$ for $n \geq 2$ and showed that $W_1(S)$ is the open unit disc when $1 \leq 1+h < \sqrt{2}$ and is the closed disc $D(0, \frac{(1+h)^2}{2[h(2+h)]^{1/2}})$ when $1+h \geq \sqrt{2}$.

Now $W_q(S)$ and $R_q(S)$ are disc for each q and in particular $R_0(S) = W_0(S) = D(0, 1+h)$ and $r_q(S) \geq w_q(S) \geq 1 + (1-q^2)h$ when $0 < |q| < 1$, but $W_q(S)$ and $R_q(S)$ are not fully known. Neither the closedness nor the radii.

Consider the weighted bilateral shift $Be_0 = ke_1$ and $Be_n = e_{n+1}$ for $n \neq 0$. Then [9] $W_1(B)$ is the open unit disc if $0 \leq k \leq 1$, and $w_1(B) > 1$ if $k > 1$. Notice that $W_q(B)$ and $R_q(B)$ are disc. But $W_q(B)$ and $R_q(B)$ are not fully known, even $w_1(B)$ when $k > 1$.

Suppose that S is a weighted shift, either unilateral or bilateral, where $\{s_n\}$ is a periodic sequence of period r. As before we can assume that $s_n \geq 0$, $n = 1, \ldots, r$. It is known [9] that $w_1(S)$ is the Perron root of the nonnegative symmetric matrix

$$S' = \frac{1}{2}\begin{pmatrix} 0 & s_1 & 0 & 0 & \cdots & s_r \\ s_1 & 0 & s_2 & 0 & \cdots & 0 \\ 0 & s_2 & 0 & & \cdots & 0 \\ & & & & & \\ \cdots & \cdots & \cdots & 0 & s_{r-2} & 0 \\ 0 & \cdots & 0 & s_{r-2} & 0 & s_{r-1} \\ s_r & 0 & \cdots & 0 & s_{r-1} & 0 \end{pmatrix}.$$

One of the striking facts about the weighted bilateral shift B with positive weights is that B is reducible if and only if its weights form a periodic sequence [3]. Ridge [7] conjectured and was recently proved that [9] $W_1(S)$ is open if and only if all the weights are nonzero. It is clear that $W_0(S) = D(0, \|S\|)$ where $\|S\| = \max_{1 \leq i \leq r} s_i$. The radius $w_q(S)$ and the closedness of $W_q(S)$ are not known when $0 < |q| < 1$. Notice that $w_1(S)$ depends on the ordering of the weights [9] when $r \geq 4$, but $w_0(S)$ is independent of the ordering. The following provides a lower bound for $w_q(S)$, which is also independent of the ordering of the weights when $|q| < 1$.

Proposition 8 *Suppose that S is a weighted shift where $\{s_n\}$ is a periodic sequence of period r, $s_n \geq 0$ for all n. Then for any $0 \leq q < 1$,*

$$r_q(S) = w_q(S) \geq \frac{1-q^2}{1-q^{2r}} \max_{\sigma \in S_r} s_{\sigma(1)} + s_{\sigma(2)}q^2 + \cdots + s_{\sigma(r)}q^{2(r-1)},$$

where S_r is the symmetric group on $\{1, \ldots, r\}$.

Proof: Suppose that S is a unilateral shift with periodic weights $s_1, \ldots, s_r$. Then construct a vector x, given $\sigma \in S_r$, such that $x_{n+kr} = \sqrt{1-q^2}q^{kr}q^{\sigma(n)-1}$, $n = 1, \ldots, r$, $k = 0, 1, \ldots$ and set $y = Ux$ where U is the unilateral shift. Notice that

$$\begin{aligned} \|x\| &= \sum_{n=1}^{\infty} x_n^2 = (1-q^2)\sum_{n=1}^{r}\sum_{k=0}^{\infty} q^{2kr}q^{2[\sigma(n)-1]} \\ &= \frac{1-q^2}{1-q^{2r}}\sum_{n=1}^{r} q^{2[\sigma(n)-1]} = \frac{1-q^2}{1-q^{2r}}\frac{1-q^{2r}}{1-q^2} = 1. \end{aligned}$$

Hence $\|y\| = 1$ and $(x, y) = q$. Thus

$$\begin{aligned} (Sx, y) &= (1-q^2)\sum_{n=1}^{r}\sum_{k=0}^{\infty} s_n q^{2kr}q^{2[\sigma(n)-1]} \\ &= \frac{1-q^2}{1-q^{2r}}\sum_{n=1}^{r} s_n q^{2[\sigma(n)-1]} \\ &= \frac{1-q^2}{1-q^{2r}}\sum_{n=1}^{r} s_{\sigma^{-1}(n)}q^{2n-2}. \end{aligned}$$

Hence we have the result when σ runs through S_r. The radius of the bilateral weighted shift is greater than or equal to the radius of the unilateral weighted shift. □

The lower bound is not equal to $w_q(S)$ in general. Otherwise by the continuity of the function $(q, A) \mapsto w_q(A)$,

$$\begin{aligned} w_1(S) &= \lim_{q\to 1}\frac{1-q^2}{1-q^{2r}}\max_{\sigma\in S_r}\sum_{n=1}^{r} s_{\sigma(n)}q^{2(n-1)} \\ &= \lim_{q\to 1}\frac{1}{1+q^2+\cdots+q^{2(r-1)}}\max_{\sigma\in S_r}\sum_{n=1}^{r} s_{\sigma(n)}q^{2(n-1)} \\ &= \frac{1}{r}\sum_{n=1}^{r} s_n. \end{aligned}$$

But $w_1(S)$ is the Perron root of the symmetric matrix S' in our previous discussion, which is generally not equal to $\frac{1}{r}\sum_{n=1}^{r} s_n$ when $r \geq 3$.

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