

Math 2660 Topics in Linear Algebra, Test 2 Fall 2008

Name:

For full credit, show all steps in details

- True or False (1 point each)
 - If $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^n$ span \mathbb{R}^n , then they are linearly independent. **True**
 - If A is an $m \times n$ matrix, then A and A^T have the same rank. **True**
 - Elementary row operations would change the row space of an $m \times n$ matrix A . **False**
 - If A is an $m \times n$ matrix, then $\dim N(A) + \dim R(A) = m$. **False**
 - \mathbf{b} is in the row space of the $m \times n$ matrix A if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent. **False**
 - If A is an $m \times n$ matrix, then the nullity of A and the nullity of A^T are the same. **False**
 - A set of $n + 1$ vectors in \mathbb{R}^n must be linearly dependent. **True**
- Show that $\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ form a basis for \mathbb{R}^2 by showing that they are linearly independent and they span \mathbb{R}^2 . (5 points)

Set $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Solving the system

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 2 & 0 \end{array} \right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & \frac{1}{2} & 0 \end{array} \right].$$

So $c_1 = c_2 = 0$ and thus the vectors are linearly independent. Now for each $\mathbf{b} = (b_1, b_2)^T \in \mathbb{R}^2$

$$\left[\begin{array}{cc|c} 2 & 3 & b_1 \\ 1 & 2 & b_2 \end{array} \right] \xrightarrow{R_2 - \frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & 3 & b_1 \\ 0 & \frac{1}{2} & b_2 - \frac{1}{2}b_1 \end{array} \right].$$

So the system is always consistent for any \mathbf{b} . Thus $\mathbf{u}_1, \mathbf{u}_2$ form a basis for \mathbb{R}^2 .

3. Let $\mathbf{u}_1 = (1, 1)^T$, $\mathbf{u}_2 = (-1, 1)^T$ and $\mathbf{v}_1 = (3, 2)^T$, $\mathbf{v}_2 = (4, 3)^T$.

- (a) Find the transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$. (3 points)
- (b) Find the transition matrix corresponding to the change of basis from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$. (2 points)
- (c) Suppose $\mathbf{x} = \mathbf{v}_1 + \mathbf{v}_2$. Find $[\mathbf{x}]_U$ where $U = \{\mathbf{u}_1, \mathbf{u}_2\}$. (2 points)

- (a) The transition matrix corresponding to the change of basis from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$S = U^{-1}V = [\mathbf{u}_1 \mathbf{u}_2]^{-1}[\mathbf{v}_1 \mathbf{v}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 5 & 7 \\ -1 & -1 \end{bmatrix}$$

- (b) The transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is $R = S^{-1} = \begin{bmatrix} -1 & -7 \\ 1 & 5 \end{bmatrix}$.

- (c) $[\mathbf{x}]_U = S[\mathbf{x}]_V = \frac{1}{2} \begin{bmatrix} 5 & 7 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix}$. For each of the following find a basis and determine the dimension.

- (a) the row space $R(A)$, (3 points)
 (b) the column space $C(A)$, (3 points)
 (c) and the null space $N(A)$. (3 points)

$$(a) R(A): A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 4R_1}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{bmatrix} \xrightarrow{\substack{-\frac{1}{5}R_2 \\ R_3 - R_2}} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\{(1, 3, 2), (0, 1, 0)\}$ is a basis for $R(A)$, the row space of A .

So $\text{rank } A = 2$.

$$(b) C(A): A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 4 & 8 \end{bmatrix} \xrightarrow{\substack{R_2 - 3R_1 \\ R_3 - 2R_1}} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{5}R_2} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The take the transpose back.

So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is a basis for $C(A)$, the column space of A .

So $\dim C(A) = 2$.

$$(c) N(A): [A|0] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 4 & 7 & 8 & 0 \end{array} \right] \xrightarrow{\substack{R_2 - 2R_1 \\ R_3 - 4R_1}} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array} \right] \xrightarrow{\substack{-\frac{1}{5}R_2 \\ R_3 - R_2}} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

So $x_3 = t$, $x_2 = 0$, $x_1 = -2t$, i.e., $N(A) = \text{span}(-2, 0, 1)^t$ and $\{(-2, 0, 1)^t\}$ is a basis for $N(A)$. So the nullity of A is 1. It matches with $\text{rank } A + \text{nullity } A = n$ where A is $m \times n$.