Math 2660 Topics in Linear Algebra, Test 2 Fall 2009 Key

Name:

For full credit, show all steps in details

1. True or False (1 point each)

   (a) If $\mathbf{x}_1, \ldots, \mathbf{x}_n \in \mathbb{R}^n$ span $\mathbb{R}^n$, then they are linearly independent. True

   (b) If A is an $m \times n$ matrix, then A and A^T have the same rank. True

   (c) Elementary row operations do not change the column space of an $m \times n$ matrix A. False

   (d) If A is an $m \times n$ matrix, then $\dim N(A) + \dim R(A) = m$. False

   (e) $\mathbf{b}$ is in the column space of the $m \times n$ matrix A if and only if the system $A\mathbf{x} = \mathbf{b}$ is consistent. True

   (f) If A is an $m \times n$ matrix, then the nullity of A and the nullity of A^T are the same. False

   (g) If A is an $m \times n$ real matrix, then $L : \mathbb{R}^n \to \mathbb{R}^m$ defined by $L(x) = Ax$, $x \in \mathbb{R}^n$, is a linear transformation. True

2. Define the kernel and range of a linear transformation $L : V \to W$. (2 points)

   The kernel of L is $\ker L = \{v \in V : L(v) = 0\}$
   The image (range) of L is $L(V) = \{L(v) : v \in V\}$.

3. Show that $\mathbf{u}_1 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ form a basis for $\mathbb{R}^2$ by showing that they are linearly independent and they span $\mathbb{R}^2$. (4 points)

   Set $c_1 \begin{bmatrix} 2 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Solving the system

   $$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ -2 & -2 & 0 \end{array}\right] \quad R_2 + R_1 \quad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

   So $c_1 = c_2 = 0$ and thus the vectors are linearly independent. Now for each $\mathbf{b} = (b_1, b_2)^T \in \mathbb{R}^2$

   $$\left[\begin{array}{cc|c} 2 & 3 & b_1 \\ -2 & -2 & b_2 \end{array}\right] \quad R_2 + R_1 \quad \left[\begin{array}{cc|c} 2 & 3 & b_1 \\ 0 & 1 & b_2 + b_1 \end{array}\right].$$

   So the system is always consistent for any $\mathbf{b}$. Thus $\mathbf{u}_1, \mathbf{u}_2$ form a basis for $\mathbb{R}^2$.

4. Let $\mathbf{u}_1 = (1,1)^T$, $\mathbf{u}_2 = (-1,1)^T$ and $\mathbf{v}_1 = (2,1)^T$, $\mathbf{v}_2 = (1,0)^T$.

(a) Find the transition matrix S corresponding to the change of basis from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$. (3 points)

(b) Find the transition matrix R corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$. (2 points)

(c) Suppose $\mathbf{x} = \mathbf{u}_1 + \mathbf{u}_2$. Find the coordinate vector $[\mathbf{x}]_V$. (2 points)

(a) The transition matrix corresponding to the change of basis from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$S = U^{-1}V = [\mathbf{u}_1\mathbf{u}_2]^{-1}[\mathbf{v}_1\mathbf{v}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix}$$

(b) The transition matrix corresponding to the change of basis from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is $R = S^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$.

(c) $[\mathbf{x}]_V = R[\mathbf{x}]_U = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, i.e., $\mathbf{x} = 2\mathbf{v}_1 - 4\mathbf{v}_2$.

5. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear operator defined by $L(x) = (-x_1, x_2)^T$. With the notations in the previous question, find

(a) the matrix representation A of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$. (3 points)

(b) the matrix representation B of L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$. (3 points)

(a) By inspection

$$\begin{aligned} L(\mathbf{u}_1) &= L(1,1)^T = (-1,1)^T = 0\mathbf{u}_1 + 1\mathbf{u}_2 \\ L(\mathbf{u}_2) &= L(-1,1)^T = (1,1)^T = 1\mathbf{u}_1 + 0\mathbf{u}_2 \end{aligned}$$

So the matrix representation of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is

$$A = [L(\mathbf{u}_1)_U \ L(\mathbf{u}_2)_U] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

(b) Since the transition matrix from $\{\mathbf{v}_1, \mathbf{v}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is S,

$$B = S^{-1}AS = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 3 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix}$$

which is the matrix representation of L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

6. Let $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix}$. For each of the following find a basis and determine rank A.

(a) the row space $R(A^T)$, (3 points)

(b) the column space $R(A)$, (3 points)

(c) and the null space $N(A)$. (3 points)

(a) Row space $R(A^T)$:

$$A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & -5 & 11 & 3 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & 4 & 3 \end{bmatrix}.$$

So

$$(1, 3, -2, 1), (0, -5, 7, 0), (0, 0, 4, 3)$$

form a basis for $R(A^T)$, the row space of A. Thus rank $A = 3$.

(b) Column space $R(A)$:

The leading ones will occur at the 1st, 2nd and third columns. So

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$$

form a basis for the column space of A.

(c) Null space $N(A)$:

$$A = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 2 & 1 & 3 & 2 & 0 \\ 3 & 4 & 5 & 6 & 0 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 0 & -5 & 7 & 0 & 0 \\ 0 & -5 & 11 & 3 & 0 \end{array}\right] \begin{matrix} \\ \\ R_3 - R_2 \end{matrix}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 0 & -5 & 7 & 0 & 0 \\ 0 & 0 & 4 & 3 & 0 \end{array}\right].$$

So $x_4 = t$, $x_3 = -\frac{3}{4}t$, $x_2 = \frac{21}{20}t$ and $x_1 = \frac{13}{20}t$, i.e., $x = t(\frac{13}{20}, -\frac{21}{20}, -\frac{3}{4}, 1)^T$. So

$$\begin{bmatrix} \frac{13}{20} \\ -\frac{21}{20} \\ -\frac{3}{4} \\ 1 \end{bmatrix}$$

is a basis for $N(A)$, and $\dim N(A) = 1$.

It matches rank $A + \text{nullity} A = n$ where A is $m \times n$.

7. (Bonus) Show that if $A, B \in \mathbb{R}^{n\times n}$ are similar, then $\det A = \det B$. (3 points)

A, B similar means that there is a nonsingular S such that $A = S^{-1}BS$, then

$$\det A = \det S^{-1} \det B \det S = \frac{1}{\det S} \det B \det S = \det B.$$