Math 2660 Topics in Linear Algebra, Test 1, Key Fall 2008

Name:

For full credit, show all steps in details

1. True or False (1 point each)

   (a) If A, B are $n \times n$ matrices and $AB = 0$, then either A is zero or B is zero. False.
   $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

   (b) Given a nonsingular matrix A, $(A^{-1})^T = (A^T)^{-1}$. True

   (c) If A is an $n \times n$ matrix such that $Ax = 0$ has only the trivial solution, then A is nonsingular. True

   (d) If A is an $n \times n$ matrix ($n \geq 2$), then $\det(cA) = c \det A$ where c is a real number. False

   (e) $\det E = c$ if E is an elementary matrix of type II corresponding to multiplying a row by c. True

   (f) $\det E = -1$ if E is an elementary matrix of type III. False

   (g) If the $n \times n$ matrices A, B are nonsingular, then AB is also nonsingular. True

   (h) If A is nonsingular, then $\det(A^{-1}) = -\det A$. False

2. Consider a linear system whose augmented matrix is of the form

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & c & 0 \end{array}\right]$$

   (a) Is it possible for the system to be inconsistent? Explain. (1 point)

   (b) For what values of c will the system have infinitely many solutions? (4 points)

   (a) Always consistent because it is homogenous and thus has the trivial solution.

   (b)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & c & 0 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & c+1 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & c-2 & 0 \end{array}\right]$$

   In order to have infinitely many solutions, $c = 2$.

3. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$. (5 points)

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] \tfrac{1}{2}R_2$$

$$\left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -6 & -3 & -2 & 0 & 1 \end{array}\right] R_3 + 6R_2 \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 6 & 1 & 3 & 1 \end{array}\right] \tfrac{1}{6}R_3 \left[\begin{array}{ccc|ccc} 1 & 4 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

$$\begin{array}{c} R_1 - 3R_3 \\ R_2 - \frac{3}{2}R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 4 & 0 & \frac{1}{2} & -\frac{3}{2} & -\frac{1}{6} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right] \begin{array}{c} R_1 - 4R_2 \\ \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{array}\right]$$

So the inverse of A is $\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{6} \end{bmatrix}$.

4. Given $A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}$.

(a) Find elementary matrices E_1, E_2, E_3 such that $E_3E_2E_1A = U$ where U is an upper triangular matrix. (2 points)

(b) Determine the inverses of E_1, E_2, E_3. What is the lower triangular matrix L such that $A = LU$? (3 points)

(a) $\begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \begin{array}{c} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{array} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} R_3 + R_2 \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = U$. So

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$L = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$.

5. Find the determinant of the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 8 & 4 \end{bmatrix}$ by using a combination of expansions and elementary row operations. (5 points)

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 2 \\ 0 & 1 & 2 & 1 \\ 1 & 1 & 8 & 4 \end{vmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ \\ R_4 - R_1 \end{matrix} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{vmatrix} = \begin{vmatrix} -3 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 7 & 3 \end{vmatrix} \begin{matrix} R_1 + 3R_2 \\ \\ \\ \end{matrix} \begin{vmatrix} 0 & 7 & 3 \\ 1 & 2 & 1 \\ 0 & 7 & 3 \end{vmatrix}$$

$$= - \begin{vmatrix} 7 & 3 \\ 7 & 3 \end{vmatrix} = 0$$