Math 2660 Topics in Linear Algebra, Quiz 8, Fall 2009 Key

Name:

For full credit, show all steps in details

1. True or False (1 point each)
   Let V be a vector space.
   (a) Different bases of the same vector space could have different number of vectors. False. Corollary 3.4.2
   (b) The dimension of the zero space is 0. True
   (c) The dimension of $\mathbb{R}^{2\times 2}$ is 4. True
   (d) Any set of n linearly independent vectors in the vector space V is a basis if $\dim V = n$. True. Theorem 3.4.3

2. Determine if the following vectors form a basis for the vector space. Give reasons.
   (a) $V = \mathbb{R}^3$ and $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}$. (3 points)
   (b) $V = \mathbb{R}^{2\times 2}$ and $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}1&0\\0&-1\end{bmatrix}$. (3 points)

(a) $2\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\1\end{bmatrix} - \begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$. So they are linearly dependent. Thus they do not form a basis.

(b) Set $c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$. So $\begin{bmatrix}c_1+c_4&c_2\\c_3&c_1-c_4\end{bmatrix} = \begin{bmatrix}0&0\\0&0\end{bmatrix}$. Hence $c_1 = c_2 = c_3 = c_4 = 0$. Thus they are linearly independent. Moreover, set

$$\begin{bmatrix}a_1&a_2\\a_3&a_4\end{bmatrix} = c_1\begin{bmatrix}1&0\\0&1\end{bmatrix}+c_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+c_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+c_4\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}c_1+c_4&c_2\\c_3&c_1-c_4\end{bmatrix}.$$

So $c_1+c_4 = a_1$, $c_2 = a_2$, $c_3 = a_3$, $c_1 - c_4 = a_4$. Hence $c_1 = (a_1+a_4)/2$, $c_2 = a_2$, $c_3 = a_3$, $c_4 = (a_1-a_4)/2$. So they span $\mathbb{R}^{2\times 2}$. Hence they form a basis for $\mathbb{R}^{2\times 2}$. Remark: You may recall $\dim \mathbb{R}^{2\times 2} = 4$ instead of showing span or linear independence.