MATH 5050/6050 Quiz 1 Fall 2009 Key

Name:

For full credit, show all steps in details

1. True or False (1 point each):

   (a) $A \in \mathbb{R}^{n\times n}$ is orthogonal if $A^{-1} = A^T$. True

   (b) Given $C \in \mathbb{R}^{n\times n}$ and $y = (y_1, \ldots, y_n)^T \in \mathbb{R}^n$. Then $y^T C = y_1 c_1^T + \cdots + y_n c_n^T$ if $c_1, \ldots, c_n$ are the columns of C. False. $c_1, \ldots, c_n$ are the columns of C^T, or equivalently $c_1^T, \ldots, c_n^T$ are the rows of C.

   (c) If the matrix $B \in \mathbb{R}^{n\times n}$ is obtained by switching the first row and the second row of $A \in \mathbb{R}^{n\times n}$, then $\det B = \det A$. False

2. The trace of $A \in \mathbb{C}^{n\times n}$ is defined as $\mathrm{Tr}\, A = \sum_{i=1}^n a_{ii}$.

   (a) Show that it is a linear function: $\mathrm{Tr}\,(\alpha A + \beta B) = \alpha \mathrm{Tr}\, A + \beta \mathrm{Tr}\, B$. (2 points)

   (b) Show that $\mathrm{Tr}\,(AB) = \mathrm{Tr}\,(BA)$ for all $A, B \in \mathbb{C}^{n\times n}$. Then show that $\mathrm{Tr}\,(PAP^{-1}) = \mathrm{Tr}\, A$ where $P \in \mathbb{C}^{n\times n}$ is nonsingular. (3 points)

(a) The diagonal entries of $\alpha A + \beta B$ are $\alpha a_{ii} + \beta b_{ii}$. So

$$\mathrm{Tr}\,(\alpha A+\beta B) = \sum_{i=1}^n (\alpha a_{ii} + \beta b_{ii}) = \sum_{i=1}^n \alpha a_{ii} + \sum_{i=1}^n \beta b_{ii} = \alpha \sum_{i=1}^n a_{ii} + \beta \sum_{i=1}^n b_{ii} = \alpha \mathrm{Tr}\, A + \beta \mathrm{Tr}\, B.$$

In other words, Tr is linear.

(b) The diagonal entries of AB are $\sum_{k=1}^n a_{ik} b_{ki}$. So

$$\mathrm{Tr}\,(AB) = \sum_{i=1}^n (\sum_{k=1}^n a_{ik} b_{ki}) = \sum_{k=1}^n (\sum_{i=1}^n a_{ik} b_{ki}) = \sum_{k=1}^n (\sum_{i=1}^n b_{ki} a_{ik}) = \mathrm{Tr}\,(BA).$$

Then $\mathrm{Tr}\,(PAP^{-1}) = \mathrm{Tr}\,(APP^{-1}) = \mathrm{Tr}\, A$.

3. Show that if $U \in \mathbb{R}^{n\times n}$ is orthogonal, then $\det U = \pm 1$. What if $U \in \mathbb{C}^{n\times n}$ is unitary? (2 points)

Since A is orthogonal, $UU^T = I_n$ so that

$$(\det U)^2 = \det U \det U^T = \det(UU^T) = \det I_n = 1.$$

Since $\det U$ is a real number as $U \in \mathbb{R}^{n\times n}$, we conclude that $\det U = \pm 1$.

When $U \in \mathbb{C}^{n\times n}$ is unitary, we have $UU^H = I_n$ so that

$$|\det U|^2 = \det(UU^H) = \det I_n = 1.$$

Thus $|\det U| = 1$.