Plotting the generalized numerical ranges associated with the compact classical groups

Tin-Yau Tam
Department of Mathematics
Auburn Univeristy, AL 36849-5307, USA
e-mail: tamtiny@mail.auburn.edu

July 30, 1996, Revised August 25, 1996

Abstract

We prove an ellitpical range theorem for the numerical range associated with $SO(3)$. The geometry behind the elliptical range theorem is explained. The result is analogous to the well-known elliptical range theorem of the C-numerical range associated with $U(2)$ or $SU(2)$. We then apply the result to give some necessary conditions of the sharp points of the C-numerical range assoicated with $SO(n)$ when $n \geq 3$, $C = e_1 e_2^T - e_2 e_1^T$, and e_1 and e_2 are the first two vectors of the standard basis of $\mathbb{R}^n$. Some properties of the generalized numerical ranges associated with the compact classical groups $SO(n)$ and $Sp(n)$ are obtained, namely, symmetry, projection, inclusion relation and direct sum. We provide MATLAB programs for plotting the numerical ranges associated with $SO(n)$ and $Sp(n)$. Various examples are discussed.

1 Introduction

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ which is equipped with a G-invariant inner product $< \cdot, \cdot >$, i.e.,

$$< \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y > = < X, Y >, \qquad X, Y \in \mathfrak{g}, \quad g \in G. \tag{1}$$

1991 Mathematics Subject Classification. Primary 15A60, 22E15
Key words and phrases. Numerical range, plotting, symplectic group, orthogonal group
Short title. Plotting the generalized numerical range

For $A_1, A_2, C \in \mathfrak{g}$, the C-numerical range of (A_1, A_2) is defined to be the following subset of $\mathbb{R}^2$:

$$W_C(A_1, A_2) = \{(< A_1, \mathrm{Ad}(g)C >, < A_2, \mathrm{Ad}(g)C >) \ : \ g \in G\}.$$

Here $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ is the adjoint representation of G. The C-numerical range $W_C(A_1, A_2)$ can be rewritten as

$$W_C(A_1, A_2) = \{(< A_1|\mathrm{Ad}^*(g)c >, < A_2|\mathrm{Ad}^*(g)c >) : g \in G\}, \tag{2}$$

where $< A|c >= c(A)$ denotes the functional value of $c \in \mathfrak{g}^*$ at the point $A \in \mathfrak{g}$ and $c \in \mathfrak{g}^*$ is dual to C, i.e., $< X, C >=< X|c >$ for all $X \in \mathfrak{g}$. The co-adjoint representation $\mathrm{Ad}^* : G \to \mathrm{Aut}(\mathfrak{g}^*)$ of G in $\mathfrak{g}^*$ is defined by $g \mapsto \mathrm{Ad}^*(g)$ such that $(\mathrm{Ad}^*(g))(c)A = c(\mathrm{Ad}(g^{-1})A)$ where $c \in \mathfrak{g}^*, A \in \mathfrak{g}$. The co-adjoint orbit of $c \in \mathfrak{g}^*$ is denoted by $G_c = \{\mathrm{Ad}^*(g)c : g \in G\}$.

The notion generalizes the classical numerical range of an $n \times n$ complex matrix A:

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, \|x\| = 1\},$$

the k-numerical range of A $(1 \le k \le n)$:

$$W_k(A) = \{\sum_{i=1}^{k} x_i^* A x_i : x_1, \ldots, x_k \quad \text{are orthonormal vectors in } \mathbb{C}^n \},$$

(The set $W_1(A)$ is the classical numerical range and $W_n(A) = \{\mathrm{tr}\, A\}$) and the c-numerical range $(c \in \mathbb{R}^n)$:

$$W_c(A) = \{\sum_{i=1}^{k} c_i x_i^* A x_i : x_1, \ldots, x_k \quad \text{are orthonormal vectors in } \mathbb{C}^n \}.$$

If $c = (1, \ldots, 1, 0, \ldots, 0)$ such that there are k copies of 1's, then $W_c(A) = W_k(A)$. One can identify $W_c(A)$ as a subset in $\mathbb{R}^2$, namely,

$$W_C(A_1, A_2) = \{(\mathrm{tr}\, A_1U^*CU, \mathrm{tr}\, A_2U^*CU) : U \in U(n)\}. \tag{3}$$

Here $A = A_1 + iA_2$ is the Hermitian decomposition of A and C is a Hermitian matrix with eigenvalues c. The Toeplitz-Hausdorff theorem [17], [6] asserts that $W(A)$ is convex. Then Halmos [5] conjectured and Berger [1] proved that $W_k(A)$ is always convex. By using Morse theory, Westwick [18] then proved that $W_c(A)$ is convex for real c and he obtained nonconvex example for complex c when $n = 3$ (the smallest n such that a nonconvex example exists). Poon [12] then gave an elementary proof. In [14] the following theorem is obtained via Atiyah's symplectic result and Kirillov's result of the symplectic structures on the co-adjoint orbits of a Lie group. It yields a class of convexity theorems which include Westwick's convexity result.

Theorem 1 *[14] Let G be a compact connected Lie group. For $A_1, A_2, C \in \mathfrak{g}$, $W_C(A_1, A_2)$ is a compact convex set in $\mathbb{R}^2$.*

Theorem 1 is best possible in the sense that if $W_C(A_1, \dots A_p) = \{(< A_1, \mathrm{Ad}(g)C >, \dots, < A_p, \mathrm{Ad}(g)C >) : g \in G\}$ where $A_1, \dots, A_p, C \in \mathfrak{g}$, then $W_C(A_1, \dots, A_p)$ fails to be convex when $p > 2$. We will see that in the next section. The underlying Lie group of the C-numerical range in (3) is the unitary group $U(n)$ if we identify the set of skew Hermitian matrices $\mathfrak{u}(n)$ with the set of Hermitian matrices $i\mathfrak{u}(n)$. Joining with $SU(n)$ (the special unitary group), $SO(n)$ (the special orthogonal group) and $Sp(n)$ (the symplectic group), they are called the compact classical group. The symplectic group $Sp(n)$ can be defined as the subgroup of $U(2n)$ of the following form:

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in U(2n).$$

Since

$$\begin{aligned} & \{(\mathrm{tr}\, A_1 U^* C U, \mathrm{tr}\, A_2 U^* C U) : U \in U(n)\} \\ = & \{(\mathrm{tr}\, A_1 U^* C U, \mathrm{tr}\, A_2 U^* C U) : U \in SU(n)\} \\ = & \{(\mathrm{tr}\, \hat{A}_1 U^* \hat{C} U, \mathrm{tr}\, \hat{A}_2 U^* \hat{C} U) : U \in SU(n)\} + \mathrm{tr}\, C(\mathrm{tr}\, A_1, \mathrm{tr}\, A_2), \end{aligned}$$

where $\hat{A}_i = A_i - \frac{\mathrm{tr}\, A_i}{n} I$, $i = 1, 2$ and $\hat{C} = C - \frac{\mathrm{tr}\, C}{n} I$, and the addition is point-wisely defined, the case of $SU(n)$ is essentially the same as the case of the $U(n)$ via a translation.

As a corollary of Theorem 1 we have

Corollary 2 *[14]*

1. *If $G = U(n)$ or $SU(n)$, then $W_C(A_1, A_2) = \{(tr\, A_1 U^* C U, tr\, A_2 U^* C U) : U \in G\}$ is convex, where A_1, A_2 and C are Hermitian matrices.*

2. *The set $W_C(A_1, A_2) = \{(tr\, A_1 O^T C O, tr\, A_2 O^T C O) : O \in SO(n)\}$ is convex, where A_1, A_2, and C are real skew symmetric matrices.*

3. *The set $W_C(A_1, A_2) = \{(tr\, A_1 O^T C O, tr\, A_2 O^T C O) : O \in O(2n+1)\}$ is convex and is equal to $\{(tr\, A_1 O^T C O, tr\, A_2 O^T C O) : O \in SO(2n+1)\}$, where A_1, A_2, and C are real skew symmetric matrices.*

4. *The set $W_C(A_1, A_2) = \{(tr\, A_1 U^* C U, tr\, A_2 U^* C U) : U \in Sp(n)\}$ is convex, where A_1, A_2, and C are in the form*

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \qquad A^* = -A, \quad B^T = B.$$

It is well known that $W_c(A)$ is an elliptical disk while $n = 2$ and $c \in \mathbb{R}^2$ (the complex $c \in \mathbb{C}^2$ case is also valid and follows from the real case) [4]. This remains true if we replace $U(2)$ by $SU(2)$ in the following

$$W_C(A_1, A_2) = \{(\operatorname{tr} A_1 U^* C U, \operatorname{tr} A_2 U^* C U) : U \in U(2)\},$$

where A_1, A_2 and C are 2×2 Hermitian matrices. Since $SU(2) = Sp(1) \cong S^3$, the unit 3-sphere in $\mathbb{R}^4$, and $SO(3) \cong S^2$, the unit 2-sphere in $\mathbb{R}^3$, it is natural to ask the shape of the numerical range assoicated with $SO(3)$. The question will be settled affirmitively in the next section.

Though we have a complete description for the numerical range associated with the low dimensional compact classical groups, namely, $SU(2)$, $U(2)$, $SO(3)$ and $Sp(1)$, we do not expect to have such nice descriptions for the generalized numerical ranges associated with the higher dimensional compact classical groups. In [9] and [11], numerical determinations of the classical numerical range are given. In [10], a computer program written in MATLAB is provided for plotting the c-numerical range, i.e., for the C-numerical range of the pair (A, B) when A, B and C are Hermitian, in which the eigenvalues of C are given by c. The same program can be applied to plot the numerical range assoicated with $SU(n)$ as well. The main idea is to generate a number of supporting lines of the convex set, whose envelope approximates the boundary of the numerical range. We will use the same idea to handle the remaining cases $SO(n)$ and $Sp(n)$. In Section 3, we will establish some basic results relying on the remarkable Kostant's convexity theorem for the compact classical groups. Basically, the study of the special orthogonal groups will be divided into two cases: $2n$ and $2n+1$. The odd case also covers $O(2n+1)$ according to Corollary 2 (3). The case of $Sp(n)$ is similar to that of $SO(2n+1)$. In Section 4, we give some inclusion relation among the numerical ranges. In Section 5, we give necessary conditions of the sharp points of the numerical range associated with $SO(n)$ when $n \geq 3$ and $C = e_1 e_2^T - e_2 e_1^T$, as an application of the elliptical range theorem in Section 2. In Section 6, we apply the results of Section 3 to offer MATLAB programs for plotting the numerical range associated with $SO(n)$ and $Sp(n)$ respectively. Needless to say, the computer generating pictures will enhance our understanding of the numerical ranges. A question concerning the center of mass of the generalized numerical range associated with $SO(2n)$ is asked.

2 Elliptical range theorem and $SO(3)$

We first establish the following lemma.[1]

Lemma 3 *Let A be an $m \times n$ real matrix and let k be the rank of A. Let S^{n-1} be the unit sphere in $\mathbb{R}^n$.*

1. *If (a) $m < n$ or (b) $k < \min\{m, n\}$, then $A(S^{n-1})$ is a $(k-1)$-ellipsoid (with interior) embedding in $\mathbb{R}^m$.*

2. *If $k = n \leq m$, then $A(S^{n-1})$ is an $(n-1)$-ellipsoid embedding in $\mathbb{R}^m$.*

Proof: The singular value decomposition [7] says that A can be decomposed as $A = USV$ where U and V are orthogonal matrices with sizes $m \times m$ and $n \times n$. The matrix S is an $m \times n$ real "diagonal" matrix whose first k diagonal entries are $s_1, \ldots, s_k$ and zeros elsewhere. Those s's are known as singular values of A.

1. We first deal with 1(a), i.e., $m < n$. Since orthogonal matrices leave the unit sphere invariant,

$$\begin{aligned}
A(S^{n-1}) &= U(\operatorname{diag}(s_1, \ldots s_k, 0, \ldots, 0)|O)V(S^{n-1}) \\
&= U(\operatorname{diag}(s_1, \ldots s_k, 0, \ldots, 0)|O)(S^{n-1}) \\
&= U(\{y \in \mathbb{R}^m : \frac{y_1^2}{s_1^2} + \ldots + \frac{y_k^2}{s_k^2} \leq 1,\ y_{k+1} = \ldots = y_m = 0\}).
\end{aligned}$$

Here O is the $m \times (n-m)$ zero matrix and k can be m. Now $U(y_1, \ldots, y_k, 0, \ldots, 0)^T = U(y_1e_1 + \ldots + y_ke_k) = y_1Ue_1 + \ldots + y_kUe_k$, where $e_1, \ldots, e_m$ are the standard orthonormal basis of $\mathbb{R}^m$. Since U is an orthogonal matrix, $Ue_1, \ldots, Ue_m$ form an orthonormal basis. With respect to this basis, we have a $(k-1)$-ellipsoid (with interior) embedding in $\mathbb{R}^m$. This just demonstrates that an orthogonal matrix transforms an ellipsoid into an ellipsoid and leaves the unit sphere invariant.

In view of 1(a), it suffices to consider $k < \min\{m, n\} = n$ for 1(b). Notice that

$$\begin{aligned}
A(S^{n-1}) &= U(\operatorname{diag}(s_1, \ldots s_k, 0, \ldots, 0)|O)^TV(S^{n-1}) \\
&= U(\operatorname{diag}(s_1, \ldots s_k, 0, \ldots, 0)|O)^T(S^{n-1}) \\
&= U(\{y \in \mathbb{R}^m : \frac{y_1^2}{s_1^2} + \ldots + \frac{y_k^2}{s_k^2} \leq 1,\ y_{k+1} = \ldots = y_m = 0\}).
\end{aligned}$$

[1]When dealing with the geometric properties of linear transformation, most standard texts on Linear Algebra are used to give examples on the images of straight lines, rectangles, etc. The result can be used to enhance the teaching of geometric properties of linear transformations, especially while $n = 2$ and $n = 3$.

So we have the same conclusion.

2. Now assume that $k = n \le m$. Then we have

$$\begin{aligned} A(S^{n-1}) &= U(\operatorname{diag}(s_1, \ldots s_n)|O)^T V(S^{n-1}) \\ &= U(\operatorname{diag}(s_1, \ldots s_n)|O)^T (S^{n-1}) \\ &= U(\{y \in \mathbb{R}^m : \frac{y_1^2}{s_1^2} + \ldots + \frac{y_n^2}{s_n^2} = 1,\ y_{n+1} = \ldots = y_m = 0\}). \end{aligned}$$

This is an $(n-1)$-ellipsoid embedding in $\mathbb{R}^m$. □

The above geometric argument has been used in [3] for the explanation of the elliptical range theorem of the classical numerical range of a 2×2 matrix when $n = 3$, i.e., S^2 is corresponding sphere. Through a translation and a rotation, the elliptical range theorem of the C-numerical range associated $G = U(2)$ [4] can be deduced from the classical case. Moreover an elliptical disk type theorem is obtained in [15] via the same argument when $n = 4$, i.e., the corresponding sphere is S^3.

The Lie algebra of $SO(3)$ consists of skew symmetric matrices. If we take the basis

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

for $\mathfrak{so}(3)$, the adjoint representation of $SO(3)$ becomes the usual representation on $\mathbb{R}^3$. The map $\varphi : \mathfrak{so}(3) \to \mathbb{R}^3$ defined as $\varphi(xE_1 + yE_2 + zE_3) = xe_1 + ye_2 + ze_3$, i.e., the coordinate map, where $e_1 = (1,0,0)^T$, $e_2 = (0,1,0)^T$ and $e_3 = (0,0,1)^T$ are the standard basis of $\mathbb{R}^3$, is an algebra isomorphism, i.e., $\varphi([X,Y]) = \varphi(X) \times \varphi(Y)$ (vector product in $\mathbb{R}^3$), where $X, Y \in \mathfrak{so}(3)$. Moreover, $\varphi(\operatorname{Ad}(g)X) = g\,\varphi(X)$, $g \in SO(3)$ (see [2], p.22). Now the nontrivial adjoint orbits can be identified, via φ, the 2-spheres centered at 0. So the numerical range $W_C(A_1, A_2)$ is just the image of the 2-sphere (i.e., $\varphi(O(C))$) of the linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ such that $Y \mapsto (\operatorname{tr} A_1 Y, \operatorname{tr} A_2 Y) = -2(\varphi(A_1) \cdot \varphi(Y), \varphi(A_2) \cdot \varphi(Y))$. Here $x \cdot y$ is the usual dot product of x and y, and $O(C) = \{\operatorname{Ad}(g)C : g \in SO(3)\}$ is the adjoint orbit of C. The image is an elliptical disk (possibly degenerated, i.e., a line or a point) centered at the origin by Lemma 3. This is the geometric explanation for the following elliptical range result when $G = SO(3)$. We now provide a complete description of the elliptical disk.

Theorem 4 *Let*

$$A_1 = \begin{pmatrix} 0 & x_1 & x_2 \\ -x_1 & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}, \ A_2 = \begin{pmatrix} 0 & y_1 & y_2 \\ -y_1 & 0 & y_3 \\ -y_2 & -y_3 & 0 \end{pmatrix}, \ C = \begin{pmatrix} 0 & c_1 & c_2 \\ -c_1 & 0 & c_3 \\ -c_2 & -c_3 & 0 \end{pmatrix} \in \mathfrak{so}(3).$$

Then $W_C(A_1, A_2) = \{(tr\, A_1 O^T C O, tr\, A_2 O^T C O) : O \in G\}$ *where* $G = SO(3)$ *or* $O(3)$, *is an elliptical disk obtained by rotating* $\mathcal{E}$ *through the angle* θ, *where* $\mathcal{E}$ *is the elliptical disk with vertices* $\pm\lambda_1\sqrt{\frac{1}{2} tr\, C^T C}$, $\pm\lambda_2\sqrt{\frac{1}{2} tr\, C^T C}$. *Here* $\lambda_1 \geq \lambda_2 \geq 0$ *and* θ *are given by the singular value decomposition of* $A = O_1 \Lambda O_2$, *where*

$$A = -2\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \end{pmatrix}, \quad \lambda_1 \geq \lambda_2 \geq 0,$$

and

$$O_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in SO(2), \quad O_2 \in O(3).$$

Hence $W_C(A_1, A_2)$ *is a nondegenerate elliptical disk (a line segment or the origin, respectively) if and only if the rank of* A *is* 2 *(*1 *or* 0*, respectively).*

Remark 1: One may obtain $\lambda_1 \geq \lambda_2$ and θ from the following spectral decomposition:

$$4G(x, y) = O_1 \mathrm{diag}\,(\lambda_1^2, \lambda_2^2) O_1^T,$$

where $G(x, y)$ is the Gram matrix of $x = (x_1, x_2, x_3)^T$ and $y = (y_1, y_2, y_3)^T$, i.e.,

$$G(x, y) = \begin{pmatrix} ||x||^2 & x \cdot y \\ x \cdot y & ||y||^2 \end{pmatrix}.$$

More explicitly,

1. if $x \cdot y = 0$, then $\lambda_1 = 2\max\{||x||^2, ||y||^2\}$, $\lambda_2 = 2\min\{||x||^2, ||y||^2\}$ and $\theta = 0$, i.e., no rotation;

2. if $x \cdot y \neq 0$, then

$$\lambda_1 = \left[2(||x||^2 + ||y||^2) + 2\sqrt{(||x||^2 - ||y||^2)^2 + 4(x \cdot y)^2}\right]^{1/2},$$
$$\lambda_2 = \left[2(||x||^2 + ||y||^2) - 2\sqrt{(||x||^2 - ||y||^2)^2 + 4(x \cdot y)^2}\right]^{1/2},$$

   and $\theta \in [-\pi/2, \pi/2]$ (or we can take $\theta + \pi$ as it will also give the same elliptical disk) is given by

$$\tan\theta = \frac{(||y||^2 - ||x||^2) + \sqrt{(||x||^2 - ||y||^2)^2 + 4(x \cdot y)^2}}{2x \cdot y},$$

   by direct computation.

Proof: Now $W_C(A_1, A_2) = \{A\varphi(Y) : Y \in O(C)\} = A(rS^2)$ where rS^2 is the 2-sphere with radius $r = \sqrt{\frac{1}{2}\mathrm{tr}\, C^T C}$, centered at the origin. Let $A = O_1 \Lambda O_2$ be the singular value decomposition of A where $O_1 \in SO(2)$ and $O_2 \in O(3)$. The matrix O_2 leaves the sphere rS^2 invariant. The matrix Λ transforms the sphere into an elliptical disk in standard position where the major axis (x-axis) is of length $2\lambda_1 r$ and the minor axis (y-axis) is of length $2\lambda_2 r$. Then O_1 just rotates the elliptical disk through the angle θ if

$$O_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

Notice that $AA^T = 4G(x, y)$ which admits the spectral decomposition. $\square$

Remark 2: The argument yields that if $C \neq 0$, $A_1, \ldots, A_p \in \mathfrak{so}(3)$, then $W_C(A_1, \ldots, A_p) = \{(\mathrm{tr}\, A_1 O^T C O, \ldots, \mathrm{tr}\, A_p O^T C_p O) : O \in SO(3)\}$ is the following set embedded into $\mathbb{R}^p$ where $p \geq 3$:

1. a nondegenerate 2-ellipsoid if $\mathrm{rank}\,(A) = 3$ (not convex);
2. an elliptical disk if $\mathrm{rank}\,(A) = 2$;
3. a line segment if $\mathrm{rank}\,(A) = 1$;
4. the origin if $\mathrm{rank}\,(A) = 0$.

Here $\mathrm{rank}\,(A)$ is the rank of the $p \times 3$ matrix whose rows are $\varphi(A_i)$, $i = 1, \ldots, p$. We may replace A by the Gram matrix of the vectors $\varphi(A_i)$, $i = 1, \ldots, p$.

3 Symmetry and projections

In this section we study the following subset in $\mathbb{R}$:

$$G(C, A) = \{\mathrm{tr}\, CU^* AU : U \in G\},$$

where A and C are in $\mathfrak{g}$ and $G = SO(n)$ or $Sp(n)$. It is clearly a compact connected set in $\mathbb{R}$, i.e., $G(C, A) = [a, b]$. In fact $G(C, A_1)$ and $G(C, A_2)$ are the projections of $W_C(A_1, A_2)$ onto the x-axis and y-axis respectively.

We denote by Σ_n the permutation group on the set $\{1, \ldots, n\}$; $G(n)$ the semi-direct product of Σ_n and $(\mathbb{Z}/2)^n$, i.e., $G(n)$ is the group of permutations φ on $\{-n, \ldots, -1, 1, \ldots, n\}$ for which $\varphi(-k) = -\varphi(k)$; $SG(n)$, the subgroup of $G(n)$ consisting of even permutations (or the semi-direct product of Σ_n and $(\mathbb{Z}/2)^{n-1}$).

We now recall the maximal tori and Weyl groups of $SO(2n+1)$, $SO(2n)$ and $Sp(n)$, respectively. The standard maximal torus of $SO(2n)$ is $T(n) = SO(2) \times \cdots \times SO(2)$ while the maximal torus of $SO(2n+1)$ is $SO(2) \times \cdots \times SO(2) \times 1$. Considering the injection $A \mapsto A \oplus 1$, we can treat $T(n) \subset SO(2n+1)$ as a maximal torus of $SO(2n+1)$. The Lie algebra $LT(n)$ of $T(n)$ is the set

$$LT(n) = \left\{ \begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \beta_n \\ -\beta_n & 0 \end{pmatrix} : (\beta_1, \ldots, \beta_n) \in \mathbb{R}^n \right\}.$$

Thus we can identify $LT(n)$ with $\mathbb{R}^n$ by sending

$$\begin{pmatrix} 0 & \beta_1 \\ -\beta_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \beta_n \\ -\beta_n & 0 \end{pmatrix} \mapsto (\beta_1, \ldots, \beta_n).$$

When $k = 2n+1$, the Weyl group W operates on the maximal torus $T(n)$ and its action is given by

$$(\beta_1, \ldots, \beta_n) \mapsto (\pm\beta_{\theta(1)}, \ldots, \pm\beta_{\theta(n)}) \tag{4}$$

where $\theta \in \Sigma_n$ and for any choice of sign. So W can be viewed as $G(n)$.

When $k = 2n$, the Weyl group W operates on the maximal torus $T(n)$ and its action is given by (4) where $\theta \in \Sigma_n$ and the number of negative signs is even. In other words, W can be viewed as $SG(n)$.

We have a canonical inclusion $U(n) \to Sp(n)$, where

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & \overline{A} \end{pmatrix} = A + 0j.$$

If we set $T^n \subset Sp(n)$ be the image of the group of diagonal unitary matrices, $\triangle(n) \subset U(n)$ ($\triangle(n)$ is a maximal torus of $U(n)$), under this inclusion, T^n is a maximal torus in $Sp(n)$. The Weyl group of $Sp(n)$ is also $G(n)$. The group $G(n)$ operates on $T^n = \triangle(n) = T(n)$ where we identify these tori using the inclusions $Sp(n) \supset U(n) \subset SO(2n)$ (see [2], p.173).

Now we are going to determine a and b by using the following results in [13].

Lemma 5 *[13] Let A be a $k \times k$ real skew symmetric matrix. Then the set $D(A) = \{diag\, U^T A U[1, 3, \ldots, 2n-1|2, 4, \ldots, 2n] : U \in SO(k)\}$ is convex, where $k = 2n$ or $k = 2n+1$. More precisely, let*

$$O^T A O = \begin{cases} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} & \text{if} \quad k = 2n, \\ \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & \lambda_n \\ -\lambda_n & 0 \end{pmatrix} \oplus 0 & \text{if} \quad k = 2n+1, \end{cases}$$

for some $O \in SO(k)$, be the canonical form of A under the adjoint action of $SO(k)$. Set $\lambda = (\lambda_1, \ldots, \lambda_n)$.

1. *When $k=2n+1$, $D(A) = \overset{\wedge}{G(n)}\lambda$, i.e., the convex hull of the elements $(\pm\lambda_{\theta(1)},\ldots,\pm\lambda_{\theta(n)})$, with $\theta\in\Sigma_n$ and any choice of sign.*

2. *When $k=2n$, $D(A) = S\overset{\wedge}{G(n)}\lambda$, i.e., the convex hull of the elements $(\pm\lambda_{\theta(1)},\ldots,\pm\lambda_{\theta(n)})$, with $\theta\in\Sigma_n$ and an even number of negative signs.*

The following result is for $O(k)$. Convexity is valid for $k=2n+1$ but fails to be true when $k=2n$. When $k=2n+1$, the result is identical with that of $SO(2n+1)$.

Lemma 6 *[13] Let A be a $k\times k$ real skew symmetric matrix. Let*

$$D(A)=\{diag\, U^TAU[1,3,\ldots,2n-1|2,4,\ldots,2n] : U\in O(k)\},$$

and

$$O^TAO=\begin{cases}\begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\oplus\cdots\oplus\begin{pmatrix}0&\lambda_n\\-\lambda_n&0\end{pmatrix} & \text{if}\quad k=2n,\\ \begin{pmatrix}0&\lambda_1\\-\lambda_1&0\end{pmatrix}\oplus\cdots\oplus\begin{pmatrix}0&\lambda_n\\-\lambda_n&0\end{pmatrix}\oplus 0 & \text{if}\quad k=2n+1,\end{cases}$$

for some $O\in O(k)$, be the canonical form of A under the adjoint action of $O(k)$. Set $\lambda=(\lambda_1,\ldots,\lambda_n)$ and $\lambda'=(\lambda_1,\ldots,\lambda_{n-1},-\lambda_n)$.

1. *If $k=2n+1$, then $D(A)$ is convex and $D(A) = \overset{\wedge}{G(n)}\lambda$, i.e., the convex hull of the elements $(\pm\lambda_{\theta(1)},\ldots,\pm\lambda_{\theta(n)})$, with $\theta\in\Sigma_n$ and any choice of sign.*

2. *If $k=2n$, then $D(A)$ is the union of $S\overset{\wedge}{G(n)}\lambda$ and $S\overset{\wedge}{G(n)}\lambda'$. ($S\overset{\wedge}{G(n)}\lambda$ is the convex hull of the elements $(\pm\lambda_{\theta(1)},\ldots,\pm\lambda_{\theta(n)})$, with $\theta\in\Sigma_n$ and even number of negative signs. $S\overset{\wedge}{G(n)}\lambda'$ is the convex hull of the elements $(\pm\lambda_{\theta(1)},\ldots,\pm\lambda_{\theta(n)})$, with $\theta\in\Sigma_n$ and odd number of negative signs.) It may not be convex.*

Lemma 7 *[13] Let $C\in\mathfrak{sp}(n)$, i.e., a $2n\times 2n$ skew Hermitian matrix of the form*

$$C=\begin{pmatrix}A&-\overline{B}\\B&\overline{A}\end{pmatrix}\in\mathbb{C}_{2n\times 2n},\quad A^*=-A,\quad B^T=B.$$

Then the set $D(C)=\{diag\,(U^*CU[1,\ldots,n|1,\ldots,n]) : U\in Sp(n)\}$ is convex. More precisely, if $\pm i\lambda_1,\ldots,\pm i\lambda_n$ (λ's are real) are the eigenvalues of C, then $D(C)$ is the convex hull of the elements $(\pm i\lambda_{\theta(1)},\ldots,\pm i\lambda_{\theta(n)})$, with $\theta\in\Sigma_n$ and any choice of sign.

Theorem 8 1. *Let G be $SO(2n+1)$ or $O(2n+1)$. Let*

$$\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & a_n \\ -a_n & 0 \end{pmatrix} \oplus 0, \tag{5}$$

and

$$\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & c_n \\ -c_n & 0 \end{pmatrix} \oplus 0 \tag{6}$$

be the canonical forms of A and $C \in \mathfrak{so}(2n+1)$ respectively, under the conjugation action of $SO(2n+1)$ or $O(2n+1)$ such that $a_1 \geq \cdots \geq a_n$ and $c_1 \geq \cdots \geq c_n$. Then $G(C, A) = [-b, b]$ where $b = 2\sum_{i=1}^n c_i a_i$.

2. *Let G be $SO(2n)$. If $n = 1$, then $G(C, A) = \{-2ac\}$ where*

$$A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}.$$

If $n \geq 2$, let

$$\begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & a_n \\ -a_n & 0 \end{pmatrix}, \tag{7}$$

and

$$\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & c_n \\ -c_n & 0 \end{pmatrix}. \tag{8}$$

be the canonical forms of A and $C \in \mathfrak{so}(2n)$ respectively, under the conjugation action of $SO(2n)$ such that $a_1 \geq \cdots \geq a_{n-1} \geq |a_n|$ and $c_1 \geq \cdots \geq c_{n-1} \geq |c_n|$. Then

$$a = -2\sum_{i=1}^{n} c_i a_i, \tag{9}$$

and

$$b = \begin{cases} 2\sum_{i=1}^n c_i a_i & \text{if} \quad n \text{ is even,} \\ 2(\sum_{i=1}^{n-1} c_i a_i - c_n a_n) & \text{if} \quad n \text{ is odd.} \end{cases} \tag{10}$$

3. *Let $G = Sp(n)$ and let*

$$diag\,(ia_1, \ldots, ia_n, -ia_1, \ldots, -ia_n),$$

and

$$diag\,(ic_1, \ldots, ic_n, -ic_1, \ldots, -ic_n)$$

be the canonical forms of A, $C \in \mathfrak{sp}(n)$, respectively, under the conjugation action of $Sp(n)$ such that $a_1 \geq \cdots \geq a_n \geq 0$ and $c_1 \geq \cdots \geq c_n \geq 0$. Then $G(C, A) = [-b, b]$ where $b = 2\sum_{i=1}^n c_i a_i$.

Proof: 1. Without loss of generality, we can assume that A and C are in the forms (5) and (6) respectively. Notice that $G(C,A) = -2\sum_{i=1}^n c_i d_i$ when $d = (d_1, \ldots, d_n) \in D(A)$, where $D(A)$ is given in Lemma 6 (1). So $G(C,A)$ is the image of the convex set $D(A)$ under the linear functional $-2c$ such that $d \mapsto -2c \cdot d$. Hence a and b are among the images of the extreme points of $D(A)$:

$$\sum_{i=1}^n c_i(\pm a_{\theta(i)}), \qquad \theta \in \Sigma_n \text{ and any choice of sign.} \tag{11}$$

It follows that $a = -2\sum_{i=1}^n c_i a_i$ and $b = -a$.

2. Similarly, we can assume that A and C are in the forms (7) and (8) respectively. Then following the above argument, a and b are among the images of the extreme points of $D(A)$ given in Lemma 6 (2), under the linear map $-2c$, i.e., those elements in (11). But the number of negative signs has to be even. So we have[2]

$$a = \begin{cases} -2\sum_{i=1}^n c_i a_i & \text{if} \quad a_n c_n \geq 0, \\ -2(\sum_{i=1}^{n-1} c_i a_i - |c_n a_n|) & \text{if} \quad a_n c_n < 0, \end{cases}$$

i.e., (9), and

$$b = \begin{cases} 2\sum_{i=1}^n c_i a_i & \text{if} \quad a_n c_n \geq 0 \text{ and } n \text{ is even}, \\ 2(\sum_{i=1}^{n-1} c_i a_i - |c_n a_n|) & \text{if} \quad a_n c_n \geq 0 \text{ and } n \text{ is odd}, \\ 2(\sum_{i=1}^{n-1} c_i a_i - |c_n a_n|) & \text{if} \quad a_n c_n < 0 \text{ and } n \text{ is even}, \\ 2(\sum_{i=1}^{n-1} c_i a_i + |c_n a_n|) & \text{if} \quad a_n c_n < 0 \text{ and } n \text{ is odd}, \end{cases}$$

i.e., (10).

3. The proof is similar to that of $SO(2n+1)$ with the application of Lemma 7. □

The following lemma is useful for plotting the generalized numerical range when $G = SO(n)$ and $Sp(n)$.

Lemma 9 *[14] Let G be a compact connected Lie group. If A_1, A_2 and C are in $\mathfrak{g}$, then the rotation of the set $W_C(A_1, A_2)$ through an angle θ is the set $W_C(A_1', A_2')$ where $(A_1', A_2') = (A_1 \cos\theta - A_2 \sin\theta,\ A_1 \sin\theta + A_2 \cos\theta)$.*

Unlike the cases $SU(n)$ or $U(n)$, the numerical ranges associated with $SO(n)$ and $Sp(n)$ exhibit some symmetry property.

Theorem 10 *Let A_1, A_2 and $C \in \mathfrak{g}$.*

[2]The appearance of a possible minus term is reminiscent of Thompson's famous singular values-diagonal elements inequalities [16]. Also see [13] and [15].

1. *When $G = SO(n)$ with $n > 2$ (if $n = 2$, then $W_C(A_1, A_2)$ is the singleton set $\{-2c(a_1, a_2)\}$), $O(2n+1)$ or $Sp(n)$, $W_C(A_1, A_2)$ contains the origin.*

2. *When $G = SO(2n+1)$, $O(2n+1)$ or $Sp(n)$, $W_C(A_1, A_2)$ is symmetric about the origin, i.e., if $x \in W_C(A_1, A_2)$, then $-x \in W_C(A_1, A_2)$.*

3. *When $G = SO(2n)$, $W_C(A_1, A_2)$ is symmetric about the origin if n is even or C is singular.*

Proof: 1. We only deal with the case $G = SO(2n)$ when $n > 1$ since the other cases follow from (2) and Corollary 2. Since the interval in Theorem 8 (2) contains the origin when $n > 2$, the projections of $W_C(A_1, A_2)$ onto the x-axis and the y-axis include the origin. By Theorem 1, we know that $W_C(A_1, A_2)$ is compact and convex. So the origin is contained in the numerical range.

2. When $G = SO(2n+1)$ or $O(2n+1)$, we can assume that C is in the form (6). Then $x \in W_C(A_1, A_2)$ implies that there exists $O \in G$ such that $x = (\operatorname{tr} CO^T A_1 O, \operatorname{tr} CO^T A_2 O)$. Obviously the matrix Q belongs to $SO(2n+1)$ where

$$Q = \begin{cases} P \oplus (1) & \text{if } n \text{ is even,} \\ P \oplus (-1) & \text{if } n \text{ is odd,} \end{cases}$$

and P is the following $2n \times 2n$ permutation matrix:

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{12}$$

Then

$$\begin{aligned} -x &= (\operatorname{tr}(-C)O^T A_1 O, \operatorname{tr}(-C)O^T A_2 O) \\ &= (\operatorname{tr}(QCQ^T)O^T A_1 O, \operatorname{tr}(QCQ^T)O^T A_2 O) \\ &= (\operatorname{tr} C(OQ)^T A_1 (OQ), \operatorname{tr} C(OQ)^T A_2 (OQ)) \in W_C(A_1, A_2). \end{aligned}$$

When $G = Sp(n)$, we can assume that $C = \operatorname{diag}(c_1, \ldots, c_n, -c_1, \ldots, -c_n)$. So if $x \in W_C(A_1, A_2)$, i.e., $x = (\operatorname{tr} CU^* A_1 U, \operatorname{tr} CU^* A_2 U)$ for some $U \in Sp(n)$. Notice that the Weyl group $G(n)$ operates on the maximal torus T^n by conjugation. So there exists $V \in Sp(n)$ such that $VCV^* = -C$. Hence

$$\begin{aligned} -x &= (\operatorname{tr}(-C)U^* A_1 U, \operatorname{tr}(-C)U^* A_2 U) \\ &= (\operatorname{tr}(VCV^*)U^* A_1 U, \operatorname{tr}(VCV^*)U^* A_2 U) \\ &= (\operatorname{tr} C(UV)^* A_1 (UV), \operatorname{tr} C(UV)^* A_2 (UV)) \in W_C(A_1, A_2). \end{aligned}$$

The approach also works for $SO(2n+1)$ without giving P explicitly and hence the result of $O(2n+1)$ follows as well.

3. If $G = SO(2n)$ and n is even, the matrix P in (12) is an element of $SO(2n)$ since $\det P = (-1)^n$ and $-C = PCP^T$ for we assume that C is in its canonical form (8). So if $x = (\operatorname{tr} CO^T A_1 O, \operatorname{tr} CO^T A_2 O)$, then

$$\begin{aligned} -x &= (\operatorname{tr}(-C)O^T A_1 O, \operatorname{tr}(-C)O^T A_2 O) \\ &= (\operatorname{tr}(PCP^T)O^T A_1 O, \operatorname{tr}(PCP^T)O^T A_2 O) \\ &= (\operatorname{tr} C(OP)^T C(OP), \operatorname{tr} C(OP)^T C(OP)) \in W_C(A_1, A_2). \end{aligned}$$

If C is singular and n is odd, i.e., $c_n = 0$, then the matrix

$$R = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

plays the role of P when we apply the same technique. $\square$

Remark 3: The symmetry property does not hold for $SO(2n)$ while n is odd and C is nonsingular. For an example, just consider $A_2 = 0$ and apply Theorem 8. See Figure 3a for a non-symmetric computer generated example in Section 6.

4 Inclusion relation and direct sum

From a result in [T2], it can be deduced that the convexity results of Theorem 1 for the cases $SO(2n+1)$, $SO(2n)$ and $Sp(n)$ are equivalent to the following inclusion relations.

Theorem 11 *We assume that C and $D \in \mathfrak{g}$ are both in the canonical forms in* (6), *Theorem 8 (3) and* (8), *when $G = SO(2n+1)$, $Sp(n)$ or $SO(2n)$, respectively.*

1. *Let $G = SO(2n+1)$ or $G = Sp(n)$. Then $W_D(A_1, A_2) \subset W_C(A_1, A_2)$ if*

$$\sum_{i=1}^k d_i \le \sum_{i=1}^k c_i \qquad k = 1, \ldots, n. \tag{13}$$

2. *Let $G = SO(2n)$ with $n \ge 2$. Then $W_D(A_1, A_2) \subset W_C(A_1, A_2)$ if*

$$\sum_{i=1}^k |d_i| \le \sum_{i=1}^k |c_i|, \qquad k = 1, \ldots, n, \tag{14}$$

$$\sum_{i=1}^{n-1} d_i - |d_n| \le \sum_{i=1}^{n-1} c_i - |c_n|, \tag{15}$$

and in addition, if the total number of negative terms of the sequences $c = (c_1, \dots, c_n)$ *and* $d = (d_1, \dots, d_n)$ *is odd, then*

$$\sum_{i=1}^{n} |d_i| \le \sum_{i=1}^{n-1} c_i - |c_n|. \tag{16}$$

Remark 4: We denote by $\prec_w$ and $\ll$ the partial orderings given by (13), and (14)–(16), respectively. It turns out that $d \prec_w c$ amounts to $d \in \hat{G(n)}c$ and $d \ll c$ amounts to $d \in \hat{SG(n)}c$ [13]. The partial orderings $\ll$ is related to the characterization of the diagonal entries of the real matrices with prescribed singular values and nonnegative determinants [16]. See [13] for recent developments. The partial ordering $\prec_w$ is often called weak majorization in the literature [7] when x and y are nonnegative vectors (Of course, the notion of weak majorization exists for all real vectors, but there is no absolute values). Theorem 10 (1) can also be deduced from Theorem 11. It is because that $0 \prec_w c$ and $0 \ll c$ for any skew symmetric C. Hence, by Theorem 11, $\{0\} = W_0(A_1, A_2) \subset W_C(A_1, A_2)$ when $G = SO(2n)$ $(n \ge 2)$, $SO(2n+1)$ and $Sp(n)$ respectively. The two partial orderings $\prec_w$ and $\ll$ play an important role for $SO(2n+1)$ ($Sp(n)$) and $SO(2n)$, respectively, as majorization plays the role for $U(n)$ or $SO(n)$. Theorem 11 is parallel to a result in [4] which is concerning majorization and the numerical range associated with $U(n)$.

It is well-known and is easy to show that for the classical numerical range,

$$W(A \oplus B) = conv\,\{W(A), W(B)\}, \tag{17}$$

where *conv* denotes the convex hull of the underlying sets.

We denote by $W(A_1, A_2)$ the C-numerical range for (A_1, A_2), where $A_1, A_2 \in \mathfrak{so}(k)$ and

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0,$$

when $G = SO(k)$ and $k \ge 2$. This is the simplest nontrivial case for $G = SO(k)$. It is also comparable with the classical numerical range which is the simplest case among the C-numerical range for $G = U(n)$. The classical numerical range can be viewed as the collection of the $(1,1)$ entry of U^*AU as U runs through the unitary group $U(n)$, i.e., the collection of (x_1, x_2) where x_1 and x_2 are the $(1,1)$ entries of U^*A_1U and U^*A_2U, respectively, if $A = A_1 + iA_2$. In the similar fashion, the numerical range $W(A_1, A_2)$ can be viewed as the collection of the order pairs $-2(x_1, x_2)$ where x_1 and x_2 are the $(1,2)$ entries of O^TA_1O and O^TA_2O, respectively, as O runs through $SO(n)$.

Motivated by (17), we want to see whether it is true true that $W(A_1 \oplus B_1, A_2 \oplus B_2) = conv\{W(A_1, A_2), W(B_1, B_2)\}$, if $A_1,\ A_2 \in \mathfrak{so}(n_1)$ and $B_1,\ B_2 \in \mathfrak{so}(n_2)$? The answer is no. For instance, if we consider

$$A_1 = \begin{pmatrix} 0 & x_1 \\ -x_1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & x_2 \\ -x_2 & 0 \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & y_1 \\ -y_2 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & y_2 \\ -y_2 & 0 \end{pmatrix},$$

then $W(A_1, A_2) = \{-2(x_1, x_2)\}$ and $W(B_1, B_2) = \{-2(y_1, y_2)\}$. The convex hull of $W(A_1, A_2)$ and $W(B_1, B_2)$ is the line segment with the end points $-2(x_1, x_2)$ and $-2(y_1, y_2)$, not necessarily passing through the origin. However $W(A_1 \oplus B_1, A_2 \oplus B_2)$ always contains the origin because of Theorem 10 (1). Indeed it is a polygon which is the convex hull of the four points $\pm(x_1, x_2)$, $\pm(y_1, y_2)$. But this is really one of the exceptional cases. We have the following result.

Theorem 12 *Suppose that $A_1, A_2 \in \mathfrak{so}(n_1)$ and $B_1, B_2 \in \mathfrak{so}(n_2)$ where $\min\{n_1, n_2\} \geq 2$. Then $W(A_1 \oplus B_1, A_2 \oplus B_2) = W(B_1 \oplus A_1, B_2 \oplus A_2) = conv\{\pm W(A_1, A_2), \pm W(B_1, B_2)\}$. Hence,*

1. *if $\min\{n_1, n_2\} \geq 3$, we have $W(A_1 \oplus B_1, A_2 \oplus B_2) = conv\{W(A_1, A_2), W(B_1, B_2)\}$.*
2. *if $n_2 = \min\{n_1, n_2\} = 2$ and $n_1 = \max\{n_1, n_2\} > 2$, we have $W(A_1 \oplus B_1, A_2 \oplus B_2) = conv\{W(A_1, A_2), \pm W(A_2, B_2)\}$.*
3. *if $n_1 = n_2 = 2$, $W(A, B)$ is the convex hull of the points $\pm 2(x_1, y_1)$ and $\pm 2(x_2, y_2)$.*

Proof: It is clear that $W(A_1 \oplus B_1, A_2 \oplus B_2) = W(B_1 \oplus A_1, B_2 \oplus A_2)$ and that $W(A_1, A_2)$ and $W(B_1, B_2)$ are subsets of $W(A_1 \oplus B_1, A_2 \oplus B_2)$ which is a compact convex set and is symmetric about the origin by Theorem 10. So $conv\{\pm W(A_1, A_2), \pm W(B_1, B_2)\} \subset W(A_1 \oplus B_1, A_2 \oplus B_2)$.

On the other hand, the rotation of $W(A_1 \oplus B_1, A_2 \oplus B_2)$ through an angle θ, denoted by $e^{i\theta} W(A_1 \oplus B_1, A_2 \oplus B_2)$, is

$$W((A_1 \cos\theta - A_2 \sin\theta) \oplus (B_1 \cos\theta - B_2 \sin\theta), (A_1 \sin\theta + A_2 \cos\theta) \oplus (B_1 \sin\theta + B_2 \cos\theta)),$$

by Lemma 9. Its projection onto the x-axis is $[-a, a]$ where a is twice of the spectral radius of $(A_1 \cos\theta - A_2 \sin\theta) \oplus (B_1 \cos\theta - B_2 \sin\theta)$, by Theorem 8. Hence $a = 2\max\{a_1, b_1\}$ where a_1 and b_1 are the spectral radii of $A_1 \cos\theta - A_2 \sin\theta$ and $B_1 \cos\theta - B_2 \sin\theta$, respectively. Now the projection of

$$e^{i\theta}\, conv\{\pm W(A_1, A_2), \pm W(B_1, B_2)\} = conv\{\pm e^{i\theta} W(A_1, A_2), \pm e^{i\theta} W(B_1, B_2)\}$$

onto the x-axis contains the projections of $\pm e^{i\theta}W(A_1,A_2)$ and $\pm e^{i\theta}W(B_1,B_2)$ onto the x-axis and $\pm a$ are both contained in one of the projections. So the interval $[-a,a]$ is a subset of the projection of $e^{i\theta}\, conv\,\{\pm W(A_1,A_2),\pm W(B_1,B_2)\}$ onto the x-axis. As a result, the projection of the compact convex set $W(A_1\oplus B_1, A_2\oplus B_2)$ is a subset of the projection of the compact convex set $conv\,\{\pm W(A_1,A_2),\pm W(B_1,B_2)\}$, along any direction. So we have $W(A_1\oplus B_1,A_2\oplus B_2)\subset conv\,\{\pm W(A_1,A_2),\pm W(B_1,B_2)\}$. Hence $W(A_1\oplus B_1,A_2\oplus B_2) = conv\,\{\pm W(A_1,A_2),\pm W(B_1,B_2)\}$.

If $\min\{n_1,n_2\}\geq 3$, then from Theorem 10 (2) and (3), $W(A_1,A_2)$ and $W(B_1,B_2)$ are symmetric about the origin. So we have case 1. Case 2 and Case 3 follow similarly. Hence the proof is completed. □

See Example 2 and Example 1 in Section 6 regarding Case 2 and Case 3 of Theorem 12 respectively.

Indeed, Case 3 of Theorem 12 can be extended to general skew symmetric C in the following way. It is actually a corollary of a more general result in [14] and so the proof is omitted.

Theorem 13 1. *Let A, B and $C\in\mathfrak{so}(2n)$. Suppose that there exists $O\in SO(2n)$ such that*

$$\begin{aligned} O^TAO &= \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & c_n \\ -a_n & 0 \end{pmatrix}, \\ O^TBO &= \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & b_n \\ -b_n & 0 \end{pmatrix}. \end{aligned}$$

Let

$$\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & c_n \\ -c_n & 0 \end{pmatrix}$$

be the canonical form of C under the conjugation action of $SO(2n)$. Then $W_C(A,B)$ is the convex hull of the points $-2(\sum_{i=1}^n c_i(\pm a_{\theta(i)}), \sum_{i=1}^n c_i(\pm b_{\theta(i)}))$, where $\theta\in\Sigma_n$ and the number of negative signs is even among $\pm a$'s.

2. *Let A, B and $C\in\mathfrak{so}(2n+1)$. Suppose that there exists $O\in SO(2n+1)$ such that*

$$\begin{aligned} O^TAO &= \begin{pmatrix} 0 & a_1 \\ -a_1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & c_n \\ -a_n & 0 \end{pmatrix}\oplus 0, \\ O^TBO &= \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & b_n \\ -b_n & 0 \end{pmatrix}\oplus 0. \end{aligned}$$

Let

$$\begin{pmatrix} 0 & c_1 \\ -c_1 & 0 \end{pmatrix}\oplus\cdots\oplus\begin{pmatrix} 0 & c_n \\ -c_n & 0 \end{pmatrix}\oplus 0$$

be the canonical form of C under the conjugation action of $SO(2n+1)$. Then $W_C(A,B)$ is the convex hull of the points $2(\sum_{i=1}^n c_i(\pm a_{\theta(i)}), \sum_{i=1}^n c_i(\pm b_{\theta(i)}))$, where $\theta \in \Sigma_n$ and there is no restriction on the choice of sign of $\pm a$'s.

3. *Let A, B and $C \in \mathfrak{sp}(n)$. Suppose that there exists $U \in Sp(n)$ such that*

$$\begin{aligned} U^*AU &= diag\,(a_1, \ldots, a_n, -a_1, \ldots, -a_n), \\ U^*BU &= diag\,(b_1, \ldots, b_n, -b_1, \ldots, -b_n). \end{aligned}$$

Let $diag\,(c_1, \ldots, c_n, -c_1, \ldots, -c_n)$ be the canonical form of C under the conjugation action of $Sp(n)$. Then $W_C(A,B)$ is the convex hull of the points

$$2(\sum_{i=1}^n c_i(\pm a_{\theta(i)}), \sum_{i=1}^n c_i(\pm b_{\theta(i)})),$$

where $\theta \in \Sigma_n$ and there is no restriction on the choice of sign of $\pm a$'s.

Theorem 14 *Suppose that $A_1, A_2, C \in \mathfrak{g}$ and let $A_1' = A_1 \oplus 0$, $A_2' = A_2 \oplus 0$ and $C' = C \oplus 0$ where 0 is the scalar zero.*

1. *Let $G = SO(k)$. If $k \geq 2$ is even, then $W_C(A_1, A_2) \subset W_{C'}(A_1', A_2')$. If $k \geq 3$ is odd, then $W_C(A_1, A_2) = W_{C'}(A_1', A_2')$.*

2. *Let $G = Sp(n)$. Then $W_C(A_1, A_2) = W_{C'}(A_1', A_2')$.*

Proof: 1. Suppose that $k = 2n \geq 2$ is even. If $x \in W_C(A_1, A_2)$, then there exists $U \in SO(k)$ such that $x = (\operatorname{tr} A_1 U^T C U, \operatorname{tr} A_2 U^T C U)$. Then set $V = U \oplus 0$ and we have $x = (\operatorname{tr} A_1' V^T C' V, \operatorname{tr} A_2' V^T C' V)$. So $x \in W_{C'}(A_1', A_2')$, i.e., $W_C(A_1, A_2) \subset W_{C'}(A_1', A_2')$.

Now suppose that $k = 2n+1 \geq 3$ is odd. By Lemma 9, $e^{i\theta} W(A_1, A_2) = W(A_1(\theta), A_2(\theta))$ and $e^{i\theta} W(A_1', A_2') = W(A_1'(\theta), A_2'(\theta))$. So the projection of $W(A_1(\theta), A_2(\theta))$ onto the x-axis is $[-a, a]$ where $a = \sum_{i=1}^n c_i a_i$. Here $a_1 \geq \cdots \geq a_n$ are from the canonical form of $A_1(\theta)$.

Similarly the projection of $W(A_1'(\theta), A_2'(\theta))$ onto the x-axis is $[-a', a']$ where $a' = \sum_{i=1}^{n+m} c_i a_i$ Here $a_1 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \geq a_{n+m}$ are from the canonical form of $A_1'(\theta)$. Since $a_{n+i} = 0$ as well as $c_{n+i} = 0$ for all $i = 1, \ldots, m$, we have $a = a'$. Hence the two compact convex sets have the same projection along any direction. So they are identical. This technique can be used to prove $k = 2n$ case though we have inclusion relation instead of equality.

2. The proof is similar to that of $SO(2n+1)$. ∎

For a strict inclusion in (1) of Theorem 14, see Example 3 in Section 6.

5 Sharp points of $W(A, B)$

In this section we reserve the notation $W(A, B)$ only for the numerical range of A, $B \in \mathfrak{so}(n)$ while

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus 0 \in \mathbb{R}_{n\times n}.$$

Let $\mathcal{C}$ be a compact convex set in $\mathbb{R}^2$. A point $(x, y) \in \partial\mathcal{C}$ is called a sharp point of $\mathcal{C}$ if there are angles θ_1 and θ_2 with $0 \leq \theta_1 < \theta_2 < 2\pi$ for which $x\cos\theta - y\sin\theta = \max\{(\alpha\cos\theta - \beta\sin\theta : (\alpha, \beta) \in \mathcal{C} \text{ for all } \theta \in (\theta_1, \theta_2)\}$. Intuitively, a sharp point is an extreme point at which the boundary takes an abrupt turn; a boundary point where there are nonunique tangents, or a corner [8].

It is well-known that a sharp point of the classical numerical range A is a normal eigenvalue of A [8]. The following gives a similar result for the sharp points of $W(A, B)$.

Theorem 15 *Let A, $B \in \mathfrak{so}(n)$, $n \geq 3$. Assume that $W(A, B) \neq \{0\}$, i.e., A and B are not both zero. If $(x, y) \in W(A, B)$ is a sharp point of $W(A, B)$, then there exists $O \in SO(n)$ such that*

$$O^TAO = X \oplus A_1, \qquad O^TBO = Y \oplus B_1,$$

where

$$X = \begin{pmatrix} 0 & -\frac{x}{2} \\ \frac{x}{2} & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -\frac{y}{2} \\ \frac{y}{2} & 0 \end{pmatrix},$$

and (x, y) is not in the interior of $W(A_1, B_1)$.

Proof: We first establish the result when $n = 3$. Since $(x, y) \in W(A, B)$, there exists $O \in SO(3)$ such that the top principal 2×2 submatrices of O^TAO and O^TBO are X and Y. Let

$$O^TAO = \begin{pmatrix} 0 & -\frac{x}{2} & x_2 \\ \frac{x}{2} & 0 & x_3 \\ -x_2 & -x_3 & 0 \end{pmatrix}, \qquad O^TBO = \begin{pmatrix} 0 & -\frac{y}{2} & y_2 \\ \frac{y}{2} & 0 & y_3 \\ -y_2 & -y_3 & 0 \end{pmatrix}.$$

Notice that $W(A, B) = W(O^TAO, O^TBO)$ and $(x, y) \in W(A, B)$ is a sharp point. So by Theorem 4, $W(A, B) = W(O^TAO, O^TBO)$ is a degenerated elliptical disk, i.e., the two vectors $(\frac{x}{2}, x_2, x_3)$ and $(\frac{y}{2}, y_2, y_3)$ are linearly dependent. Moreover, (x, y) is one of the vertices. This amounts to $\lambda^2 = x^2 + y^2$ where $\lambda, 0$ are the singular values of the matrix

$$K = \begin{pmatrix} -x & 2x_2 & 2x_3 \\ -y & 2y_2 & 2y_3 \end{pmatrix}.$$

But $\lambda^2 = \operatorname{tr} KK^*$ which implies that x_2, x_3, y_2 and y_3 are all zeros.

Now assume that $n > 3$. Let (x, y) be a sharp point of $W(A, B)$. Then there exists $O \in SO(n)$ such that the top principal 2×2 submatrices of $A' = O^T AO$ and $B' = O^T BO$ are X and Y. Consider a'_{ij} for some $i = 1, 2$ and $3 \le j \le n$. Let $P \in O(n)$ be the permutation matrix obtained by exchanging the jth and the third rows of the identity matrix. Then a'_{ij} is the $(i, 3)$ entry of $\hat{A} = P^T O^T AOP$. Set $\hat{B} = P^T O^T BOP$. Notice that $\det P = -1$ when $j > 3$ and $\det P = 1$ when $j = 3$, and

$$\{(\operatorname{tr} C(Q \oplus \hat{I})^T \hat{A}(Q \oplus \hat{I}), \operatorname{tr} C(Q \oplus \hat{I})^T \hat{B}(Q \oplus \hat{I})) : Q \in SO(3)\} \subset W(A, B), \tag{18}$$

where $\hat{I} = I \oplus \delta$, I is $(n-4) \times (n-4)$ identity matrix and $\delta = 1$ if $j = 3$ and $\delta = -1$ if $j > 3$. But the left side of (18) is $W(\hat{A}[1, 2, 3|1, 2, 3], \hat{B}[1, 2, 3|1, 2, 3])$ where $A[\alpha|\beta]$ denotes the submatrix of A lying on the rows and columens given by the finite sequences α and β, respectively. Hence by the case $n = 3$, $a'_{ij} = 0$, $i = 1, 2$ and $3 \le j \le n$. Similarly $b'_{i,j} = 0$, $i = 1, 2$ and $3 \le j \le n$. As a result, we have $O^T AO = X \oplus A_1$ and $O^T BO = Y \oplus B_2$ for some $O \in SO(n)$. By Theorem 12, we have $W(A, B) = W(O^T AO, O^T BO) = \{\pm W(X, Y), \pm W(A_1, B_1)\}$. Moreover (x, y) is not in the interior of $W(A_1, B_1)$ since $(x, y) \in \partial W(A, B)$. □

Remark 5: The necessary conditions of Theorem 15 are not sufficient for (x, y) being a sharp point of $W(A, B)$. It is because that (x, y) may appear on the boundary of $W(A_1, B_1)$ and $W(A_1, B_1)$ may have no sharp points at all.

However if $A = X \oplus A_1$ and $B = Y \oplus B_1$ and if $(x, y) \notin W(A_1, B_1)$, i.e., $W(X, Y) = \{(x, y)\} \not\subset W(A_1, B_1)$, then (x, y) is a sharp point of $W(A, B)$. See Example 2 in Section 6.

Remark 6: Now if we denote by $\hat{W}(A, B)$ the numerical range of A, $B \in \mathfrak{sp}(n)$ when

$$C = \operatorname{diag}(i, 0, \ldots, 0, -i, 0, \ldots, 0),$$

then $\hat{W}(A, B)$ is the collection of $-2(x, y)$ where x and y are the $(1, 1)$ entry of $U^* AU$ and $U^* BU$ as U runs through $Sp(n)$. We can derive some necessary condition for the sharp points of $\hat{W}(A, B)$, e.g., if (x, y) is a sharp point of $\hat{W}(A, B)$, then there exists $U \in Sp(n)$ such that

$$U^* AU = \begin{pmatrix} A_1 & -\overline{B}_1 \\ B_1 & \overline{A}_1 \end{pmatrix}, \qquad U^* BU = \begin{pmatrix} A_2 & -\overline{B}_2 \\ B_2 & \overline{A}_2 \end{pmatrix},$$

where

$$A_1 = (\frac{-ix}{2}) \oplus A'_1, \qquad A_2 = (\frac{-iy}{2}) \oplus A'_2,$$

and the $(n + 1, 1)$ entries of B's are zeros. It is not hard to deduce that by using elliptical range theorem for the classical numerical range and the proof is omitted.

6 Plotting the generalized numerical ranges

This section is mainly to present computer programs for generating the graphs of the numerical ranges by using MATLAB. For $G = SO(2n+1)$ or $Sp(n)$, the programs are written by using (1) and (3) of Theorem 10. However we need to take special care for the case $G = SO(2n)$. The command `[U, S] = schur(A)` is useful to compute the canonical form in (2) of Theorem 10. However, it only provides the "non-standard" decomposition $S = U^TAU = A_1 \oplus A_2 \cdots \oplus A_k$ where A's are either 1×1 zero block[3] (the total number of such 1×1 blocks is even) or 2×2 skew symmetric matrix. The determinant of U can be -1. Nevertheless, it can be remedied. Firstly, we use MATLAB to get U and the canonical form, probably "non-standard". Secondly we count p, the number of negative superdiagonal entries of the canonical form. We handle this by using the command `find`. Thirdly, we get the eigenvalues of iA, $\pm a_1, \pm a_2, \ldots, \pm a_n$ where $a_1 \geq \cdots \geq a_{n-1} \geq |a_n|$ and the sign of a_n will be determined. If $\det U$ is 1, then a_n is nonegative (nonpositive) when p is even (odd). If $\det U$ is -1, then a_n is nonegative (nonpositive) when p is odd (even).

The following program is written as an m-file for MATLAB in order to plot the C-numerical range of (A, B) associated with $G = SO(2n+1)$, $O(2n+1)$ and $SO(2n)$ where A, B, and C are real skew symmetric matrices of the same size.

```
function y = so(A,B,C)
% This is a program for plotting the C-numerical range of (A,B)
% where A, B and C are n1xn1 skew symmetric matrices, n1 > 2.
% The case n1=2 is trivial. It is a point in R^2.
if any(any(A ~= -A')) | any(any(B ~= -B')) | any(any(C ~= -C'))
error('Some matrix is not skew symmetric')      % warning for non-skew
                                                   % symmetric matrices
else
n1 = length(C);
 if n1 == 2
error('The size of the matrices has to be at least 3')
 elseif rem(n1,2) == 1
% Now we compute a number of rectangles which enclose the C-numerical
% range of (A,B).
```

[3] We note that the 1×1 blocks are very likely to appear when A is a direct sum of two skew symmetric matrix and one of which is odd dimensional. So we do not count the number of negative terms among the $(2i-1, 2i)$ entries of S, $s_{2i-1,2i}$, $i = 1, \ldots, n$ as a means to determine the sign of a_n.

```
c = sort(abs((i*(eig(C))))) ;    % arrange the moduli of the
                                 % eigenvalues of C in ascending order
for k = 1:50
    t = k*pi/100 ;
    A1 = cos(t)*A - sin(t)*B ;
    B1 = sin(t)*A + cos(t)*B ;
    a1 = sort(abs(i*(eig(A1)))) ;  % moduli of the eigenvalues of A1
    b1 = sort(abs(i*(eig(B1)))) ;  % and B1 in ascending order
    a = c'*a1 ;                    % [-a,a] and [-b,b] are the projections
    b = c'*b1 ;                    % onto the x-axis and y-axis of the
                                   % rotated W(A,B,C)
    X1 = [a a -a -a a]' ;
    Y1 = [b -b -b b b]' ;
    X(:,k) = cos(t)*X1 + sin(t)*Y1 ;   % X and Y contain the x- and y-
    Y(:,k) = -sin(t)*X1 + cos(t)*Y1 ;  % coordinates of the vertices of
                                       % the rectangles
end
% Hence we plot all the rectangles whose opposite sides
% are the supporting line segments so that the enevople will
% be the C-numerical range of (A,B).
    plot(X,Y,'r') ;
title('The C-numerical range of (A,B) associated with SO(2n+1) amd O(2n+1)')
xlabel('x-axis')
ylabel('y axis')
pause ;
grid ;
pause ;
y = ' YOU HAVE JUST SEEN THE C-NUMERICAL RANGE OF (A,B) ' ;
else
n = (n1)/2 ;
e = (10)^(-10) ;
[U,schurC] = schur(C) ;     % Schur form of C,  det(U) can be -1
x1 = real(diag(schurC,1)) ;
m = length(find(x1<e) ;     % m is the number of negative eigenvalues
             % appearing on the superdiagional of the Schur form of C
```

```
   if round(real(det(U))) == -1    % det (U) = -1
   m = m-1 ;
   end
m ;             % m even means that c_n \ge 0; odd means that c_n < 0
for k = 1:50
    t = k*pi/100 ;
    A1 = cos(t)*A - sin(t)*B ;
    B1 = sin(t)*A + cos(t)*B ;
    [V,schurA1] = schur(A1) ;
y1 = real(diag(schurA1,1)) ;
j = length(find(y1<e)) ;
   if round(real(det(V))) == -1
   j = j-1 ;
   end
j ;           % j even means that a_n \ge 0; odd means that a_n < 0
   [W,schurB1] = schur(B1) ;
z1 = real(diag(schurB1,1)) ;
l = length(find(z1<e)) ;
   if round(real(det(W))) == -1
   l = l-1 ;
   end
l ;             % l even means that b_n \ge 0; odd means that b_n < 0
        % With respect to the closed fundamental Weyl chamber, we have
     % a_1 \ge ... \ge a_{n-1} \ge |a_n|. Similar for C, i.e., c_1 \ge ...
     % c_{n-1} \ge |c_n| and B, i.e., b_1 \ge ... \ge b_{n-1} \ge |b_n|
   c1 = -sort(-abs(eig(C))) ;           % in descending order
   a11 = -sort(-abs(eig(A1))) ;
   MA1 = c1'*a11 ;              % MA1 = 2*sum_1^n |a_ic_i|
   mA1 = -MA1 ;                     % mA1 = -2*sum_1^n |a_ic_i|
   a11(2*n-1) = -a11(2*n-1) ;
   a11(2*n) = -a11(2*n) ;
   MA2 = c1'*a11 ;    % MA2 = 2*(sum_1^{n-1} |a_ic_i| - |a_nc_n|)
   mA2 = -MA2 ;            % mA2 = -2*(sum_1^{n-1} |a_ic_i| - |a_nc_n|)
if rem(m+j,2) == 0           % a_nc_n \ge 0
   a1 = mA1 ;                % [a1,a2] is the projection
```

```
        if  rem(n-1,2) == 0             % n is odd
        a2 = MA2 ;
        else                            % n is even
        a2 = MA1 ;
        end
else                            % a_nc_n < 0
   a1 = mA2 ;
        if rem(n-1,2) == 0      % n is odd
        a2 = MA1 ;
        else                    % n is even
        a2 = MA2 ;
        end
end
   b11 = -sort(-abs(eig(B1))) ;
   MB1 = c1'*b11 ;           % MB1 = 2*sum_1^n |b_ic_i|
   mB1 = -MB1 ;                  % mB1 = -2*sum_1^n |b_ic_i|
   b11(2*n-1) = -b11(2*n-1) ;
   b11(2*n) = -b11(2*n) ;
   MB2 = c1'*b11 ;     % MB2 = 2*(sum_1^{n-1} |b_ic_i| - |b_nc_n|)
   mB2 = -MB2 ;            % mB2 = -2*(sum_1^{n-1} |b_ic_i| - |b_nc_n|)
if rem(m+1,2) == 0          % b_nc_n \ge 0
   b1 = mB1 ;              % [b1,b2] is the projection
        if  rem(n-1,2) == 0             % n is odd
        b2 = MB2 ;
        else                            % n is even
        b2 = MB1 ;
        end
else                        % b_nc_n < 0
   b1 = mB2 ;
        if rem(n-1,2) == 0         % n is odd
        b2 = MB1 ;
        else                       % n is even
        b2 = MB2 ;
        end
end
```

```
    X1 = [a1 a1 a2 a2 a1]' ;
    Y1 = [b1 b2 b2 b1 b1]' ;
    X(:,k) = cos(t)*X1 + sin(t)*Y1 ;
    Y(:,k) = -sin(t)*X1 + cos(t)*Y1 ;
end
% Hence we plot all the rectangles whose opposite sides
% are the supporting line segments so that the enevople will
% be the C-numerical range of (A,B).
    plot(X,Y,'r') ;
title('The C-numerical range of (A,B) associated with SO(2n)')
xlabel('x-axis')
ylabel('y axis')
pause ;
grid ;
pause ;
y = ' YOU HAVE JUST SEEN THE C-NUMERICAL RANGE OF (A,B) ' ;
end
end
```

The following program is written for MATLAB for plotting the C-numerical range of (A, B) where A, B, and C are matrices in $i\mathfrak{sp}(n)$. We notice that $W_C(A, B) = -W_{iC}(iA, iB)$ and the numerical range is symmetric. So it makes no difference if we assume that $A, B, C \in i\mathfrak{sp}(n)$ instead of $\mathfrak{sp}(n)$.

```
function y = sp(A,B,C)
% This is a program for plotting the C-numerical range of (A,B)
% where A, B and C are in i*sp(n), of size 2nx2n.
% They are of the form
%                      [A1 A2^-
%                       A2  -A1^-]
% where A1 is nxn Hermitian and A2 is nxn symmetric.
n1 = length(A) ;
n2 = length(B) ;
n3 = length(C) ;
if rem(n1,2) ~= 0 | rem(n2,2) ~= 0 | rem(n3,2) ~= 0
error('Some matrix is not even dimensional')
```

```
elseif any(any(A ~= A'))|any(any(B ~= B'))|any(any(B ~= B'))
error('Some matrix is not Hermitian')
else
   n = n1/2 ;
   for i = 1:n
      for j = 1:n
   A11(i,j) = A(i,j) ;
   A21(i,j) = A(i+n,j) ;
   A22(i,j) = A(i+n,j+n) ;
   B11(i,j) = B(i,j) ;
   B21(i,j) = B(i+n,j) ;
   B22(i,j) = B(i+n,j+n) ;
   C11(i,j) = C(i,j) ;
   C21(i,j) = C(i+n,j) ;
   C22(i,j) = C(i+n,j+n) ;
   end
      end
end
if A22 == -conj(A11) & B22 == -conj(B11)    % CAUSTION!
& C22 == -conj(C11) &  A21 == conj((A21)')  % use only one line
& B21 == conj((B21)') & C21 == conj((C21)') % for the if statement
% Now we generate a number of rectangles which enclose the C-numerical
% range of (A,B).
c = sort(abs((eig(C)))) ;       % arrange the moduli of the
                                % eigenvalues of C in ascending order
for k = 1:50
    t = k*pi/100 ;
    A1 = cos(t)*A - sin(t)*B ;
    B1 = sin(t)*A + cos(t)*B ;
    a1 = sort(abs(eig(A1))) ;    % moduli of the eigenvalues of A1 and
    b1 = sort(abs(eig(B1))) ;    % B1 in ascending order
    a = c'*(a1) ;                % [-a,a] and [-b,b] are the projections
    b = c'*(b1) ;                % onto the x-axis and y-axis of the
                                 % rotated W(A,B,C)
    X1 = [a a -a -a a]' ;
```

```
    Y1 = [b -b -b b b]' ;
    X(:,k) = cos(t)*X1 + sin(t)*Y1 ;   % X and Y contain the x- and y-
    Y(:,k) = -sin(t)*X1 + cos(t)*Y1 ;  % coordinates of the vertices of
                                       % the rectangles
end
% Hence we plot all the rectangles whose opposite sides are
% the supporting line segments so that the enevople will be
% the C-numerical range of (A,B).
    plot(X,Y,'r') ;
title('The C-numerical range of (A,B) associated with Sp(n)')
xlabel('x-axis')
ylabel('y-axis')
pause ;
grid ;
pause ;
y = ' YOU HAVE JUST SEEN THE C-NUMERICAL RANGE OF (A,B) ' ;
else
error('Some matrix is not in i*LSp(n)')
end
end
```

We apply the programs for plotting the numerical ranges $W_C(A_1, A_2)$ of the matrices in the following examples.

Example 1:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 0 & 0 & -3 & 2 \\ 0 & 0 & -1 & 1 \\ 3 & 1 & 0 & 2 \\ -2 & -1 & -2 & 0 \end{pmatrix}.$$

The underlying group $SO(2n)$ is $SO(4)$ and $n = 2$ is even. So by Theorem 10, the shape in Figure 1 is symmetric about the origin. Since the skew symmetric matrices A and B are simultanteously put into their canonical forms under the conjugation action of $SO(4)$, it is a polygon. See Theorem 13 (1).

Example 2:

$$A = \begin{pmatrix} 0 & 6 & 7 \\ -6 & 0 & 1 \\ -7 & -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 3 & -9 \\ -3 & 0 & -2 \\ 9 & 2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The underlying group $SO(2n+1)$ is $SO(3)$. By Theorem 8, it is an elliptical disk since the two matrices A and B are linearly independent. See Figure 2a.

Set

$$A' = A \oplus \begin{pmatrix} 0 & -10 \\ 10 & 0 \end{pmatrix}, \qquad B' = B \oplus \begin{pmatrix} 0 & 10 \\ -10 & 0 \end{pmatrix}.$$

Notice that the point $\pm(20,-20) \notin W(A,B)$ which is an elliptical disk by Theorem 4. So $\pm(20,-20)$ are sharp points of $W(A',B') = conv\,\{\pm W(A,B), \pm(20,-20)\}$. See Figure 2b.

Example 3:

$$A = \begin{pmatrix} 0 & -1 & 3 & 5 \\ 1 & 0 & -1 & 2 \\ -3 & 1 & 0 & 1 \\ -5 & -2 & -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & -7 \\ 7 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 8 & 6 \\ 1 & 0 & 2 & -1 \\ -8 & -2 & 0 & -2 \\ -6 & 1 & 2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

By Theorem 10, we remark that $n = 3$ is the second smallest integer such that a non-symmetric figure exists for the numerical range associated with $SO(2n)$. The smallest integer is $n = 1$ which gives a singleton set. See Figure 3a. If we set $A' = A \oplus 0$, $B' = B \oplus 0$ and $C' = C \oplus 0$ where 0 is the scalar zero (the odd case now), then $W_{C'}(A',B')$ is shown in Figure 3b. Notice that $W_C(A,B) \subset W_{C'}(A',B')$ according to Theorem 14.

Example 4:

$$A = \begin{pmatrix} 15 & 12 & 12 \\ 12 & 13 & 8 \\ 12 & 8 & 18 \end{pmatrix} \oplus \begin{pmatrix} -15 & -12 & -12 \\ -12 & -13 & -8 \\ -12 & -8 & -18 \end{pmatrix}, \quad B = \begin{pmatrix} 6 & 14 & 5 \\ 14 & 17 & 15 \\ 5 & 15 & 11 \end{pmatrix} \oplus \begin{pmatrix} -6 & -14 & -5 \\ -14 & -17 & -15 \\ -5 & -15 & -11 \end{pmatrix},$$

and

$$C = \mathrm{diag}\,(1,1,1,-1,-1,-1).$$

See Figure 4 for the numerical range.

Example 5:

$$A = \begin{pmatrix} 14 & 0 & 0 \\ 0 & -6 & 5-7i \\ 0 & 5+7i & 2 \end{pmatrix} \oplus \begin{pmatrix} -14 & 0 & 0 \\ 0 & 6 & -5-7i \\ 0 & -5+7i & -2 \end{pmatrix},$$

$$B = \begin{pmatrix} 18 & 0 & 0 \\ 0 & 8 & -6-19i \\ 0 & -6+19i & 18 \end{pmatrix} \oplus \begin{pmatrix} -18 & 0 & 0 \\ 0 & -8 & 6-19i \\ 0 & 6+19i & -18 \end{pmatrix},$$

and

$$C = \mathrm{diag}\,(1, 2, 1-1, -2, -1).$$

See Figure 5 for the numerical range.

It is certainly true that when $G = SO(2n+1)$, $Sp(n)$, and $SO(2n)$ (if n is even), the center of mass (assume uniform density) of $W_C(A, B)$ is the origin, due to the symmetry property. It is also true when C is singular by Theorem 10.

It would be interesting to find the center of mass of the numerical range associated with $G = SO(2n)$ when $n \geq 3$ is odd. When $n = 1$, the numerical range is a singleton set. Based on some computer generated examples, we specalute that the origin is always the center of mass when $n \geq 3$ is odd.

References

[1] C. A. Berger, *Normal Dilations*, Cornell University, Ph. D dissertation (1963).

[2] T. Bröcker and T. tom Dieck, *Representations of compact Lie groups*, New York: Springer-Verlag, 1985.

[3] C. Davis, The Toeplitz-Hausdorff theorem explained, Canad. Math. Bull. **14** (2), 1971, 245-246.

[4] M. Goldberg and E. G. Straus, Elementary inclusion relations for generalized numerical ranges, **18**, (1977), 1-24.

[5] P. Halmos, *A Hilbert Space Problem Book*, Springer-Verlag, New York, 1978.

[6] F. Hausdorff, Der Wertvorrat einer Bilinearform, Math Z. **3** (1919), 314-316.

[7] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, 1990.

[8] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[9] C. R. Johnson, Numerical determination of the field of values of a general complex matrix, SIAM J. Num. Analysis, **15** (1978), 595-602.

[10] C. K. Li, C. H. Sung and N. K. Tsing, The c-convex matrices: characterizations, inclusion relations and normality, Linear and Multilinear Algebra, **25** (1989), 275-287.

[11] M. Marcus and C. Pesce, Computer generated numerical ranges and some resulting theorems, Linear and Multlinear Algebra, **20** (1987), 121-157.

[12] Y. T. Poon, Another proof of a result of Westwick, Linear and Multilinear Algebra, **9** (1980), 35-37.

[13] T.Y. Tam, Kostant's convexity theorem and classical compact groups, to appear in Linear and Multilinear Algebra.

[14] T.Y. Tam, Generalized numerical ranges, numerical radii, and Lie groups, preprint, 1996.

[15] T.Y. Tam, Miranda-Thompson's inequality and a log convexity result, to appear in Linear Algebra and Its Applications.

[16] R. Thompson, Singular values, diagonal elements, and convexity, SIAM J. Appl. Math. **32** (1977), 39-63.

[17] O. Toeplitz, Das algebraishe Analogon zu einem Satze von Fejér, Math. Z. **2** (1918), 187-197.

[18] R. Westwick, A theorem on numerical range, Linear and Multilinear Algebra **2** (1975), 311-315.