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PFAFFIAN AND DECOMPOSABLE NUMERICAL RANGE OF A COMPLEX SKEW SYMMETRIC MATRIX

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ABSTRACT. In the literature it is known that the decomposable numerical range $W_k^\wedge(A)$ of $A \in \mathbb{C}_{n\times n}$ is not necessarily convex. But it is not known whether $W_k^\wedge(A)$ is star-shaped. We construct a symmetric unitary matrix $A \in \mathbb{C}_{n\times n}$ such that the decomposable numerical range $W_k^\wedge(A)$ is not star-shaped and hence not simply connected. We then consider a real analog $R_k^\wedge(A)$ and show that $R_k^\wedge(A)$ is star-shaped if $A \in \mathbb{C}_{n\times n}$ is skew symmetric. Such star-shapedness result is also true for the Pfaffian numerical range $P_k^\wedge(A)$.

1. INTRODUCTION

Let $\mathbb{C}_{n\times n}$ be the set of $n \times n$ complex matrices. Given $A \in \mathbb{C}_{n\times n}$, the classical numerical range of A is the compact set

$$W(A) := \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\},$$

which is the image of the (compact) unit sphere $\mathbb{S}^{n-1} \subset \mathbb{C}^n$ under the nonlinear map $x \mapsto x^*Ax$. Toeplitz-Hausdorff theorem asserts that $W(A)$ is a convex set. For a simple proof see [13]. When $n = 2$, $W(A)$ is an elliptical disk (possibly degenerated) [7], known as the elliptical range theorem.

Among many generalizations of the numerical range $W(A)$, one is given in the context of multilinear algebra. Given $1 \le k \le n$, the kth decomposable numerical range of A [9, 10] is defined to be the following set

$$W_k^\wedge(A) = \{\det((U^*AU)[k|k]) : U \in \mathrm{U}(n)\}, \tag{1.1}$$

where $\mathrm{U}(n)$ is the unitary group in $\mathbb{C}_{n\times n}$ and $B[k|k]$ denotes the $k \times k$ principal submatrix of $B \in \mathbb{C}_{n\times n}$ lying in the first k rows and the first k columns. Evidently $W_1^\wedge(A) = W(A)$ and $W_n^\wedge(A) = \{\det A\}$. We remark that in the formulation (1.1) the unitary group $\mathrm{U}(n)$ can be

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replaced by the special unitary group $\mathrm{SU}(n)$ and the set remains the same:

$$W_k^\wedge(A) = \{\det((U^*AU)[k|k]) : U \in \mathrm{SU}(n)\}, \tag{1.2}$$

The kth decomposable numerical range can be written in the context of the kth exterior space $\wedge^k\mathbb{C}^n$ [10]. Let $x^\wedge := x_1 \wedge \cdots \wedge x_k$ and $y^\wedge := y_1 \wedge \cdots \wedge y_k$ be two decomposable vectors in $\wedge^k\mathbb{C}^n$. The (standard) inner product in $\mathbb{C}^n$ induces an inner product on $\wedge^k\mathbb{C}^n$:

$$(x^\wedge, y^\wedge) = \det((x_i, y_j)).$$

The kth compound $C_k(A)$ of A is the operator on $\wedge^k\mathbb{C}^n$ such that

$$C_k(A)x_1 \wedge \cdots \wedge x_k = Ax_1 \wedge \cdots \wedge Ax_k.$$

It is known that [10]

$$W_k^\wedge(A) = \{(C_k(A)x^\wedge, x^\wedge) : x_1, \ldots, x_k \in \mathbb{C}^n \text{ are orthonormal }\}. \tag{1.3}$$

Clearly $W_k^\wedge(A)$ is always convex if $k = 1$ or $k = n$. When $k = 1$, it is simply the classical case $W(A)$. When $k = n$, $W_n^\wedge(A) = \{\det A\}$. It is also known that $W_{n-1}^\wedge(A)$ is also convex [9]. It is due to the fact that when $k = n - 1$, all vectors in the exterior space $\wedge^{n-1}\mathbb{C}^n$ are decomposable so that $W_{n-1}^\wedge(A) = W(C_{n-1}(A))$. Hence $W_{n-1}^\wedge(A)$ is convex. Indeed every element of $\wedge^k\mathbb{C}^n$ is decomposable if and only if $k = 1$ or $k = n - 1$ [9, Lemma 3]. However for $1 < k < n - 1$, $W_k^\wedge(A)$ is not convex [9, 14] in general. See the following example and more discussion will be given in the next section.

Example 1.1. Consider the complex unitary symmetric matrix

$$A = \mathrm{diag}\,(i, i, 1, \ldots, 1) \in \mathbb{C}_{n\times n},$$

where $n \geq 4$. Let $1 < k < n - 1$. It is known that $\pm 1 \in W_k^\wedge(A)$ but $0 \notin W_k^\wedge(A)$ [9, 14]. So it is not convex.

In the above example the matrix $A \in \mathbb{C}_{n\times n}$ is very nice: symmetric and unitary, but convexity still does not hold (of course the symmetric property is not invariant under unitarily similarity). In the literature it is not known whether $W_k^\wedge(A)$ is star-shaped for general $A \in \mathbb{C}_{n\times n}$. Indeed we will show that the above example is star-shaped in the next section. However this is not the case for general $A \in \mathbb{C}_{n\times n}$. We will construct a symmetric and unitary $A \in \mathbb{C}_{n\times n}$ such that $W_k^\wedge(A)$ is not star-shaped (thus it is not simply connected). So the star-shapedness result does not hold for $W_k^\wedge(A)$, unlike the C-numerical range $W_C(A)$ of $A \in \mathbb{C}_{n\times n}$ [5] and other related generalizations [17, 3, 4].

The real analog of the numerical range has also been studied [1], that is, if the domain $\mathbb{S}^{n-1}$ is replaced by $\mathbb{S}^{n-1}_{\mathbb{R}} := \mathbb{S}^{n-1} \cap \mathbb{R}^n$:

$$R(A) := \{x^T A x : x \in \mathbb{R}^n, x^T x = 1\} \subset W(A).$$

Since $x^T B x = 0$ for $x \in \mathbb{R}^n$ and skew symmetric $B \in \mathbb{C}_{n\times n}$ (i.e., $B^T = -B$), we have

$$x^T A x = x^T(\frac{A + A^T}{2})x + x^T(\frac{A - A^T}{2})x = x^T(\frac{A + A^T}{2})x.$$

Thus

$$R(A) = R(\frac{A + A^T}{2})$$

but in general

$$W(A) \neq W(\frac{A + A^T}{2}).$$

It is known that if $A \in \mathbb{C}_{n\times n}$ with $n \geq 3$, then [1, 12] $R(A) = W(\frac{A+A^T}{2})$ and hence is convex, and $R(A)$ is an ellipse (possibly degenerate) when $n = 2$.

Given $A \in \mathbb{C}_{n\times n}$, as $R(A)$ is the real analog of $W(A)$, we now introduce a real analog of $W^\wedge_k(A)$. For $1 \leq k \leq n$, define the compact set,

$$\begin{aligned} (1.4)\quad R^\wedge_k(A) &:= \{\det((O^T A O)[k|k]) : O \in \mathrm{SO}(n)\} \\ &= \{\det((O^T A O)[k|k]) : O \in \mathrm{O}(n)\} \\ &= \{\det(X^T A X) : X \in \mathbb{R}_{n\times k} \text{ with o.n. columns }\} \\ &\subset W^\wedge_k(A), \end{aligned}$$

where $\mathrm{O}(n)$ is the orthogonal group and $\mathrm{SO}(n)$ is the special orthogonal group. Alike the groups $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ giving the same $W^\wedge_k(A)$, $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ yield the same $R^\wedge_k(A)$.

Similar to the complex case, one has $R^\wedge_1(A) = R(A)$ and $R^\wedge_n(A) = \{\det A\}$. Moreover

$$R^\wedge_{n-1}(A) = R(C_{n-1}(A))$$

and hence is convex if $n \geq 3$. However, unlike $R(A)$, $R^\wedge_k(A) \neq R^\wedge_k(\frac{A+A^T}{2})$ for $k \geq 2$ and general $A \in \mathbb{C}_{n\times n}$.

Since $R^\wedge_k(A)$ is a subset of $W^\wedge_k(A)$, it is natural to ask if convexity may hold for $R^\wedge_k(A)$ even though $W^\wedge_k(A)$ is not necessarily convex (see Example 1.1), hoping that the "problematic points" in $W^\wedge_k(A)$ would go away. However Example 1.1 gives a negative answer. We will construct an example $R^\wedge_k(A)$ which is not star-shaped. Indeed it is the very same non-star-shaped example for $W^\wedge_k(A)$.

In this paper we also prove that if $A \in \mathbb{C}_{n\times n}$ is skew symmetric, i.e., $A = -A^T$, then $R_k^\wedge(A)$ is star-shaped with respect to the origin. The proof involves a notion called *Pfaffian decomposable numerical range*. We will first give a brief review on Pfaffian of a skew symmetric matrix and then introduce the Pfaffian decomposable numerical range in Section 3. Then in Section 4 we obtain the star-shapedness results. Some convexity results are obtained in Section 5. Finally in Section 6 we make a remark on the decomposable numerical range under congruence.

2. $W_k^\wedge(A)$ IS NOT STAR-SHAPED

The main result of this section is to construct a matrix $A \in \mathbb{C}_{n\times n}$ such that the decomposable $W_k^\wedge(A)$ and its real analog $R_k^\wedge(A)$ are not star-shaped for some $1 < k < n-1$. We first give a full description of Example 1.1 which turns out to be star-shaped with respect to i.

Example 2.1. Let $1 < k < n-1$. Consider the complex unitary symmetric matrix

$$A = \mathrm{diag}\,(i, i, 1, \ldots, 1) \in \mathbb{C}_{n\times n},$$

where $n \geq 4$. To completely describe the set $W_k^\wedge(A)$, write

$$A = \frac{e^{-i\pi/4}}{\sqrt{2}}(\mathrm{diag}\,(-1,-1,1,\ldots,1) + iI_n).$$

Now

$$\det((U^*AU)[k|k]) = \frac{e^{-ik\pi/4}}{2^{k/2}}\det[(U^*\mathrm{diag}\,(-1,-1,1,\ldots,1)U)[k|k]+iI_k].$$

If $1 < k < n-1$, the eigenvalues of the following $k \times k$ submatrix of $U^*\mathrm{diag}\,(-1,-1,1,\ldots,1)U$

$$(U^*\mathrm{diag}\,(-1,-1,1,\ldots,1)U)[k|k]$$

has eigenvalues $1,\ldots,1,\alpha,\beta$ where α,β range over $[-1,1]$ according to the interlacing inequalities for submatrix of a Hermitian matrix. We remark that the result for the real case O^TAO ($O \in \mathrm{O}(n)$) is also valid. Thus

$$\begin{aligned} & R_k^\wedge(A) \\ = \;& W_k^\wedge(A) \\ = \;& \{\frac{e^{-ik\pi/4}}{2^{k/2}}(1+i)^{k-2}(\alpha+i)(\beta+i) : \alpha,\beta \in [-1,1]\} \\ = \;& \{\frac{e^{-i\pi/2}}{2}(\alpha\beta - 1 + i(\alpha+\beta)) : \alpha,\beta \in [-1,1]\}. \end{aligned}$$

So

$$R_k^{\wedge}(A) = W_k^{\wedge}(A) = \frac{e^{-i\pi/2}}{2} S,$$

where

$$S := \{\alpha\beta - 1 + i(\alpha + \beta) : \alpha, \beta \in [-1, 1]\}.$$

Now consider the map $\gamma : \mathbb{R}^2 \to \mathbb{R}^2$

$$(\alpha, \beta) \mapsto (\alpha\beta - 1, \alpha + \beta).$$

If $\alpha + \beta = c$, where $c \in [-2, 2]$, then

$$\alpha\beta = \alpha(c - \alpha) = -(\alpha - \frac{c}{2})^2 + (\frac{c}{2})^2.$$

So

$$\max_{\alpha+\beta=c} \alpha\beta - 1 = (\frac{c}{2})^2 - 1,$$

and is attainable at $\alpha = \beta = \frac{c}{2}$;

$$\min_{\alpha+\beta=c} \alpha\beta - 1 = \begin{cases} c - 2 & \text{if } 0 \le c \le 2 \\ -c - 2 & \text{if } -2 \le c < 0 \end{cases}$$

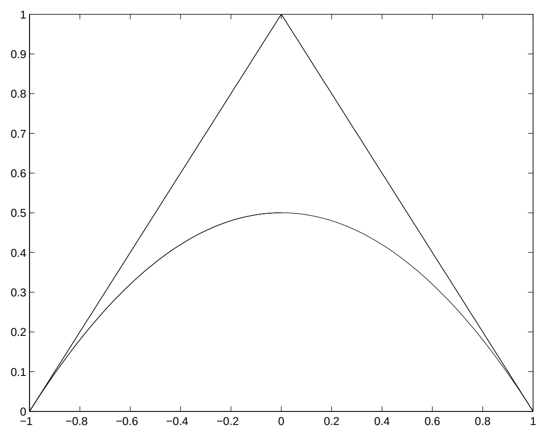
and is attainable at

$$\alpha \text{ or } c - \alpha = \begin{cases} 1 & \text{if } 0 \le c \le 2 \\ -1 & \text{if } -2 \le c < 0. \end{cases}$$

Thus S is the region bounded by the parabola $x = (\frac{y}{2})^2 - 1$ and the lines $x = y - 2$ and $x = -y - 2$.

Hence if $1 < k < n - 1$, then $R_k^{\wedge}(A) = W_k^{\wedge}(A)$ is the region R in $\mathbb{C}$ bounded by the parabola $x^2 = -2y + 1$ and the lines $y = x + 1$ and $y = -x + 1$ (see the figure below) so that it is star-shaped with star center i.

Plot of the region R



We make an interesting observation: the shape of $R_k^\wedge(A) = W_k^\wedge(A)$ is independent of the choice of k, where $1 < k < n-1$.

Based on Example 2.1, we are going to construct a non-star-shaped example for $R_k^\wedge(A)$ and $W_k^\wedge(A)$.

Example 2.2. Let $A = \operatorname{diag}(1,1,1,1,1,1,i,i,i,i)$. We claim that $W_6^\wedge(A)$ is not star-shaped. Write

$$A = \frac{e^{-i\pi/4}}{\sqrt{2}}(\operatorname{diag}(1,\dots,1,-1,-1,-1,-1) + iI_n).$$

Now

$$\begin{aligned} &\det((U^*AU)[k|k]) \\ = \ &\frac{e^{-ik\pi/4}}{2^{k/2}}\det[(U^*\operatorname{diag}(1,\dots,1,-1,-1,-1,-1)U)[k|k] + iI_k]. \end{aligned}$$

If $4 \le k < n-1$, the eigenvalues of the $k \times k$ submatrix

$$(U^*\operatorname{diag}(1,\dots,1,-1,-1,-1,-1)U)[k|k]$$

of $U^*\operatorname{diag}(1,\dots,1,-1,-1,-1,-1)U$ has eigenvalues $1,\dots,1,\alpha,\beta,\gamma,\delta$ where $\alpha,\beta,\gamma,\delta$ range over $[-1,1]$ according to the interlacing inequalities for submatrix of a Hermitian matrix. Thus

$$\begin{aligned} W_k^\wedge(A) \ &= \ \{\frac{e^{-ik\pi/4}}{2^{k/2}}(1+i)^{k-2}(\alpha+i)(\beta+i)(\gamma+i)(\delta+i) : \\ &\qquad \alpha,\beta,\gamma,\delta \in [-1,1]\} \end{aligned}$$

which clearly does not contain the origin. Let $B := A[8|8]$. Notice that

$$R = W_6^\wedge(B) \subset W_6^\wedge(A)$$

where R is the region given in Example 2.1. On the other hand

$$-R = i^2 W_4^\wedge(B) \subset W_6^\wedge(A).$$

So $(-R) \cup R \subset W_6^\wedge(A)$ but $W_6^\wedge(A)$ does not contain the origin. So $W_6^\wedge(A)$ is not star-shaped.

Remark 2.3. Notice that $R_6^\wedge(A) = W_6^\wedge(A)$ so that $R_6^\wedge(A)$ is not star-shaped as well.

3. Pfaffian numerical range of a skew symmetric matrix

Let $A = (a_{ij}) \in \mathbb{C}_{2n\times 2n}$ be a skew symmetric matrix. The Pfaffian of A is defined as

$$\operatorname{Pf}(A) = \frac{1}{2^n n!}\sum_{\sigma\in S_{2n}} \operatorname{sgn}(\sigma)\prod_{i=1}^n a_{\sigma(2i-1),\sigma(2i)},$$

where S_{2n} is the symmetric group of degree $2n$ and $\operatorname{sgn}(\sigma)$ is the signature of σ. For example

$$\operatorname{Pf}\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a, \quad \operatorname{Pf}\begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

It is known that [6] for $A, B \in \mathbb{C}_{2n\times 2n}$ where A is skew symmetric, then

(1) $\operatorname{Pf}(A)^2 = \det(A)$,
(2) $\operatorname{Pf}(BAB^T) = (\det B)\operatorname{Pf}(A)$,
(3) $\operatorname{Pf}(\lambda A) = \lambda^n \operatorname{Pf}(A)$,
(4) $\operatorname{Pf}(A^T) = (-1)^n \operatorname{Pf}(A)$.

The Pfaffian of a skew-symmetric matrix $A \in \mathbb{C}_{(2n+1)\times(2n+1)}$ is defined to be zero, as the determinant of A is zero. The Pfaffian is an invariant polynomial of a skew-symmetric matrix under a special orthogonal basis change. It is important in the theory of characteristic classes. See [8, 18] for some applications of Pfaffian.

The Pfaffian of a skew symmetric $B \in \mathbb{C}_{2k\times 2k}$ can be computed recursively as

$$\operatorname{Pf}(B) = \sum_{i=2}^{2k} (-1)^i b_{1i} \operatorname{Pf}(B_{1i}), \tag{3.1}$$

where $B_{1i} \in \mathbb{C}_{(2k-2)\times(2k-2)}$ denotes the submatrix of B obtained by removing the first and the ith rows and the first and the ith columns. The Pfaffian of the 0×0 matrix is equal to one by convention.

Let $A \in \mathbb{C}_{n\times n}$ be skew symmetric. Then each element of $\{O^T AO : O \in \mathrm{SO}_n\}$ is skew symmetric. For an even integer $2k \le n$, we introduce

$$P^{\wedge}_{2k}(A) := \{\operatorname{Pf}(O^T AO[2k|2k]) : O \in \mathrm{SO}_n\} \tag{3.2}$$

and call $P^{\wedge}_{2k}(A)$ the $2k$-*Pfaffian numerical range* of A.

Recall $R^{\wedge}_{2k}(A) = \{\det((O^T AO)[2k|2k]) : O \in \mathrm{SO}(n)\}$. Since $\det B = \operatorname{Pf}(B)^2$ for skew symmetric $B \in \mathbb{C}_{2n\times 2n}$,

$$R^{\wedge}_{2k}(A) = \{z^2 : z \in P^{\wedge}_{2k}(A)\}.$$

We simply write

$$R^{\wedge}_{2k}(A) = [P^{\wedge}_{2k}(A)]^2 \tag{3.3}$$

in which the square is defined element-wise, i.e.,

$$C^2 := \{z^2 : z \in C\} \subset \mathbb{C}$$

for any subset $C \in \mathbb{C}$. It turns out in the next section that $P^\wedge_{2k}(A)$ is star-shaped with respect to the origin. Consequently, $R^\wedge_{2k}(A)$ is also star-shaped.

Remark 3.1. If we replace $\mathrm{SO}(n)$ by $\mathrm{O}(n)$, then everything in (3.2) remains the same unless $2k = n$. One would have $\{\pm \mathrm{Pf}\,(A)\}$ instead of the singleton set $\{\mathrm{Pf}\,(A)\}$ because of the formula $\mathrm{Pf}\,(OAO^T) = (\det O)\mathrm{Pf}\,(A)$. Nevertheless, the $2k = n$ case is trivial.

4. Star-shapedness and skew symmetric matrices

We first establish a star-shapedness result for the Pfaffian numerical range $P^\wedge_{2k}(A)$, where $2 \le 2k \le n$ and $A \in \mathbb{C}_{n\times n}$ is skew symmetric.

Theorem 4.1. Let $A \in \mathbb{C}_{n\times n}$ be a complex skew symmetric matrix.
(1) If n is even, then $P^\wedge_n(A) = \{\mathrm{Pf}\,A\}$.
(2) If $2k < n$, then $P^\wedge_{2k}(A)$ is star-shaped with respect to the origin.

Proof. (1) When $2k = n$,

$$\mathrm{Pf}\,(O^T AO) = (\det O)\mathrm{Pf}\,(A) = \mathrm{Pf}\,(A)$$

if $O \in \mathrm{SO}(n)$. Thus $P^\wedge_n(A) = \{\mathrm{Pf}\,(A)\}$.

(2) Now assume that $2k < n$. Notice that

$$P^\wedge_{2k}(A) = \{\mathrm{Pf}\,(X^T AX) : X \in \mathbb{R}_{n\times 2k} \text{ with orthonormal columns } \}.$$

Suppose $\alpha \in P^\wedge_{2k}(A)$. We are going to show that the line segment $[0, \alpha] \subset P^\wedge_{2k}(A)$. We will construct an ellipse $E_0 \subset P^\wedge_{2k}(A)$ containing α and centered at the origin, and a line segment $[\gamma, -\gamma] \subset P^\wedge_{2k}(A)$. Moreover we will show that E_0 continuously deforms into $[\gamma, -\gamma]$ within $P^\wedge_{2k}(A)$. Thus $[0, \alpha] \in P^\wedge_{2k}(A)$.

Let $X = [x_1 \cdots x_{2k}] \in \mathbb{R}_{n\times 2k}$ where $x_1, \ldots, x_{2k} \in \mathbb{R}^n$ are orthonormal such that $\alpha = \mathrm{Pf}\,(X^T AX)$. Notice that $X^T AX = (x_i^T Ax_j) \in \mathbb{C}_{(2k-2)\times(2k-2)}$ so that

$$\alpha = \mathrm{Pf}\,(X^T AX) = \mathrm{Pf}\,(x_i^T Ax_j).$$

Since $2k < n$, let $x_{2k+1} \in \mathbb{R}^n$ be a unit vector orthogonal to $x_1, \ldots, x_{2k}$. Let

$$x_1(\theta) := (\cos\theta)x_1 + (\sin\theta)x_{2k+1}$$

and

$$X_\theta := [x_1(\theta)\ x_2 \cdots x_{2k}].$$

Clearly $X_0 = X$. By the recursive formula (3.1)

$$\begin{aligned}
\xi(\theta) &:= \mathrm{Pf}\,(X_\theta^T A X_\theta) \\
&= \sum_{i=2}^{2k} (-1)^i ([(\cos\theta)x_1 + (\sin\theta)x_{2k+1}]^T A x_i)\mathrm{Pf}\,([X_\theta^T A X_\theta]_{1i}) \\
&= (\cos\theta)\sum_{i=2}^{2k} (-1)^i (x_1^T A x_i)\mathrm{Pf}\,([X^T A X]_{1i}) \\
&\qquad +(\sin\theta)\sum_{i=2}^{2k} (-1)^i (x_{2k+1}^T A x_i)\mathrm{Pf}\,([X^T A X]_{1i}) \\
&= (\cos\theta)\alpha + (\sin\theta)\beta \in P_{2k}^{\wedge}(A),
\end{aligned}$$

where $\alpha = \mathrm{Pf}\,(X_0^T A X_0)$ and $\beta := \mathrm{Pf}\,(X_{\pi/2}^T A X_{\pi/2}) \in \mathbb{C}$. The locus

$$E_0 := \{\xi(\theta) : \theta \in [0, 2\pi]\} \subset P_{2k}^{\wedge}(A)$$

is an ellipse (possibly degenerate) contained in $P_{2k}^{\wedge}(A)$. The origin is enclosed by the ellipse. If the ellipse degenerates, then the result follows.

We claim that

$$0 \in P_{2k}^{\wedge}(A),$$

i.e., there is $Y \in \mathbb{R}_{n\times 2k}$ with orthonormal columns such that $\mathrm{Pf}\,(Y^T AY) = 0$. We are going to establish the claim.

Case 1: $2k < n-1$. Let $y_1 \in \mathbb{R}^n$ be any unit vector. Since $2k < n-1$, if we write $Ay_1 = u + iv$ where $u, v \in \mathbb{R}^n$, there are orthonormal vectors $y_2, \cdots, y_{2k} \in \mathbb{R}^n$ orthogonal to y_1, u, v. Hence there is $Y := [y_1 \ \cdots \ y_{2k}] \in \mathbb{R}_{n\times 2k}$ with orthonormal columns such that the first row and the first column of $Y^T AY$ are zero vectors.

Case 2: $2k+1 = n$. Notice that the skew symmetric $A \in \mathbb{C}_{n\times n}$ has a zero eigenvalue. So there is a nonzero vector $y \in \mathbb{C}^n$ such that $Ay = 0$. Write

$$y = \xi_1 y_1 + \xi_2 y_2,$$

where $y_1, y_2 \in \mathbb{R}^n$ are orthonormal. Extend y_1, y_2 to an orthonormal set $\{y_1, \ldots, , y_{2k}\}$ in $\mathbb{R}^n$. Since $Ay = 0$, Ay_1 and Ay_2 are linearly dependent and hence $\det(Y^T AY) = 0$, where $Y := [y_1 \ \cdots \ y_{2k}] \in \mathbb{R}_{n\times 2k}$. Thus $\mathrm{Pf}\,(Y^T AY) = 0$.

The claim is now proved, i.e., there is $Y \in \mathbb{R}_{n\times 2k}$ with orthonormal columns such that $\mathrm{Pf}\,(Y^T AY) = 0$. Let $Y = [y_1 \cdots y_{2k}]$ and let $y_{2k+1} \in \mathbb{R}^n$ be a unit vector orthogonal to $y_1, \ldots, y_{2k}$. Set

$$y_1(\theta) := (\cos\theta)y_1 + (\sin\theta)y_{2k+1}$$

and

$$Y_\theta := [y_1(\theta)\ y_2 \cdots y_{2k}].$$

Similar to the previous computation, we have

$$\eta(\theta) = \mathrm{Pf}\,(Y_\theta^T A Y_\theta) = (\sin\theta)\gamma \in P_{2k}^\wedge(A),$$

where $\gamma := \mathrm{Pf}\,(Y_{\pi/2}^T A Y_{\pi/2})$. The locus

$$E_1 := \{\eta(\theta) : \theta \in [0, 2\pi]\} \subset P_{2k}^\wedge(A)$$

is a line segment $[-\gamma, \gamma]$ containing the origin. Since the orthogonal group is path connected, there is a continuous path $Z(t) \in \mathbb{R}_{n\times 2k}$ ($t \in [0,1]$) with orthonormal columns, such that $Z(0) = X$ and $Z(1) = Y$. Consider the continuous function

$$\begin{aligned} \varphi(t,\theta) : &= \mathrm{Pf}\,((Z(t)R(\theta))^T AZ(t)R(\theta)) \\ &= \mathrm{Pf}\,(R(\theta)^T Z(t)^T AZ(t)R(\theta)). \end{aligned}$$

Notice that $E_0 = \{\varphi(0,\theta) : \theta \in [0, 2\pi]\}$ and $E_1 = \{\varphi(1,\theta) : \theta \in [0, 2\pi]\}$. So for each point $\xi \in [0, \alpha]$ there is some $t \in [0,1]$ such that

$$\xi \in E_t := \{\varphi(t,\theta) : \theta \in [0, 2\pi]\} \subset P_{2k}^\wedge(A).$$

Hence the star-shapedness result is established.

□

Now we use Theorem 4.1 to establish the star-shapedness of the decomposable numerical range $R_{2k}^\wedge(A)$.

Theorem 4.2. Let $A \in \mathbb{C}_{n\times n}$ be a complex skew symmetric matrix. Then $R_k^\wedge(A)$ is star-shaped with respect to the origin for all $1 \le k \le n$. More precisely,

(1) $R_n^\wedge(A) = \{\det A\}$.
(2) $R_{2k+1}^\wedge(A) = \{0\}$, when $1 \le 2k+1 \le n$.
(3) $R_{2k}^\wedge(A)$ is star-shaped with respect to the origin, where $1 < 2k \le n$.

Proof. (1) Obvious.

(2) Notice that the compound matrix $C_{2k+1}(A)$ is skew symmetric since

$$C_{2k+1}(A)^T = C_{2k+1}(A^T) = C_{2k+1}(-A) = -C_{2k+1}(A).$$

Thus $(C_{2k+1}(A)x^\wedge, x^\wedge) = 0$ for all $x^\wedge \in \wedge^{2k+1}\mathbb{C}^n$ and the desired result follows from (1.3).

(3) By (3.3)

$$R_{2k}^\wedge(A) = (P_{2k}^\wedge(A))^2.$$

Set $C := P_{2k}^\wedge(A)$. We need to show that if $x \in C$, then the line segment $[0, x^2]$ joining 0 and x^2 is contained in $C^2 = R_{2k}^\wedge(A)$. We may assume that $x \neq 0$. Notice that $(e^{i\theta}C)^2 = e^{i2\theta}C^2$ for any $\theta \in [0, 2\pi]$, i.e., rotating C by the angle θ would rotate C^2 by 2θ. Moreover the property of star-shapedness with star center 0 remains invariant under rotation. So we may assume that $x > 0$. By Theorem 4.1, $[0, x] \in C$. So $[0, x^2] \in C^2 = R^{\wedge}_{2k}(A)$.

□

5. Some Convexity Results

In this section we show that $P^{\wedge}_{2k}(A)$ is convex if $k = 1$ or $2k + 1 = n$, where $A \in \mathbb{C}_{n\times n}$ is skew symmetric (convexity of $R^{\wedge}_{2k}(A)$ is known for $2k + 1 = n$ and general $A \in \mathbb{C}_{n\times n}$).

Theorem 5.1. Let $A \in \mathbb{C}_{n\times n}$ be a complex skew symmetric matrix. Then

(a) $P^{\wedge}_2(A)$ is convex.
(b) If $n = 2k + 1$, then $P^{\wedge}_{2k}(A)$ is convex.
(c) When $n = 3$, $P^{\wedge}_2(A)$ is an elliptical disk centered at the origin and $R^{\wedge}_2(A)$ is an elliptical disk (not necessarily centered at the origin). In particular, if

$$A = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix},$$

then

$$P^{\wedge}_2(A) = \{(x, y, z) \cdot w : w \in \mathbb{S}^2_{\mathbb{R}}\}$$

where $\mathbb{S}^2_{\mathbb{R}}$ is the unit sphere in $\mathbb{R}^3$ and $s \cdot t := s_1t_1 + s_2t_2 + s_3t_3$ denotes the standard inner product of $\mathbb{R}^3$, and

$$R^{\wedge}_2(A) = R(\hat{A}), \quad \hat{A} := \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix}.$$

Proof. (a) Without loss of generality we may assume that $2 < n$. It is because that if $n = 2$, then $P^{\wedge}_2(A) = \{\mathrm{Pf}(A)\}$ and $R^{\wedge}_2(A) = \{\det A\}$; both are convex.

For all $x \in \mathbb{R}^n$, $x^TAx = 0$ since A is complex skew symmetric. For $x_1, x_2 \in \mathbb{R}^n$, $x_1^TAx_2 = -x_2^TAx_1$ so that

$$\begin{aligned} (C_2(A)x_1 \wedge x_2, x_1 \wedge x_2) &= \det \begin{bmatrix} x_1^TAx_1 & x_1^TAx_2 \\ x_2^TAx_1 & x_2^TAx_2 \end{bmatrix} \\ &= \det \begin{bmatrix} 0 & x_1^TAx_2 \\ x_2^TAx_1 & 0 \end{bmatrix} \\ &= (x_1^TAx_2)^2 \in \mathbb{C}. \end{aligned}$$

Write $A = A_1 + iA_2$ where $A_1, A_2 \in \mathbb{R}_{n\times n}$ are real skew symmetric matrices. Now

$$\begin{aligned} P_2^\wedge(A) &= \{x_1^T A x_2 : x_1, x_2 \in \mathbb{R}^n \text{ o.n.}\} \\ &= \{x_1^T A_1 x_2 + i x_1^T A_2 x_2 : x_1, x_2 \in \mathbb{R}^n \text{ o.n.}\}. \end{aligned}$$

which can be identified with the subset

$$\{(\operatorname{tr} CO^T A_1 O, \operatorname{tr} CO^T A_2 O) : O \in \mathrm{SO}(n)\} \subset \mathbb{R}^2$$

where

$$C = \frac{1}{2}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \oplus O_{n-2}.$$

By a result of Tam [16], the set $P_2^\wedge(A)$ is a convex set containing the origin.

(b) Suppose $2k+1 = n$. Let $\alpha = \mathrm{Pf}\,(X^T A X)$ and $\beta = \mathrm{Pf}\,(Y^T A Y)$ where $X, Y \in \mathbb{R}_{n\times 2k}$ have orthonormal columns. We want to show that the line segment $[\alpha, \beta]$ is contained in $P_{2k}^\wedge(A)$. We may assume that $\alpha \neq -\beta$ by Theorem 4.1. So we can assume that $\alpha \neq \pm\beta$.

Write $X = [x_1 \cdots x_{2k}]$, $Y = [y_1 \cdots y_{2k}]$. Let $U = \operatorname{span}\{x_1, \ldots, x_{2k}\}$ and $V = \operatorname{span}\{y_1, \ldots, y_{2k}\}$. By the dimension theorem

$$\begin{aligned} \dim(U \cap V) &= \dim U + \dim V - \dim(U+V) \\ &\geq 2k + 2k - n = 4k - (2k+1) = 2k-1. \end{aligned}$$

Let $w_1, \ldots, w_{2k-1} \in U \cap V$ be $2k-1$ orthonormal vectors. Extend $\{w_1, \ldots, w_{2k-1}\}$ to two orthonormal sets

$$\{w_1, \ldots, w_{2k-1}, x\}, \quad \{w_1, \ldots, w_{2k-1}, y\}$$

in $\mathbb{R}^n$ so that their spans are U and V, respectively. There are $O_x, O_y \in \mathrm{SO}(2k)$ such that

$$W_x := [w_1 \ldots w_{2k-1}\ x] = [x_1\ x_2 \ldots\ x_{2k}] O_x$$

and

$$W_y := [w_1 \ldots w_{2k-1}\ y] = [y_1\ y_2 \ldots\ y_{2k}] O_y.$$

Then by Pfaffian's property (2)

$$\begin{aligned} \mathrm{Pf}\,(W_x^T A W_x) &= \mathrm{Pf}\,((XO_x)^T A (XO_x)) \\ &= \mathrm{Pf}\,(O_x^T X^T A X O_x) \\ &= (\det O_x) \mathrm{Pf}\,(X^T A X) = \alpha. \end{aligned}$$

Similarly $\mathrm{Pf}\,(W_y^T A W_y) = \beta$.

Since $\alpha \neq \pm\beta$, x and y must be linearly independent. Let $z_1, z_2 \in \mathbb{R}^n$ be orthonormal such that $\operatorname{span}\{z_1, z_2\} = \operatorname{span}\{x, y\}$. Let

$$W_\theta := [w_1\ \cdots w_{2k-1}\ (\cos\theta) z_1 + (\sin\theta) z_2] \in \mathbb{R}_{n\times 2k}, \quad \theta \in [0, 2\pi].$$

The columns of W_θ are orthonormal since x, y are orthogonal to $w_1, \ldots, w_{2k-1}$. Using the idea of the proof of Theorem 4.1, the locus

$$\begin{aligned} L &:= \{\mathrm{Pf}\,(W_\theta^T A W_\theta) : \theta \in [0, 2\pi]\} \\ &= \{(\cos\theta)(\mathrm{Pf}\,W_0^T A W_0) + (\sin\theta)(\mathrm{Pf}\,W_{\pi/2}^T A W_{\pi/2}) : \theta \in [0, 2\pi]\} \\ &\subset P_{2k}^\wedge(A) \end{aligned}$$

is an ellipse (possible degenerate) and contains α and β and the ellipse is centered at the origin. All the points enclosed by the ellipse are in $P_{2k}^\wedge(A)$. In particular the line segment $[\alpha, \beta] \subset P_{2k}^\wedge(A)$.

(c) Let

$$A = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{bmatrix}.$$

Recall $R_2^\wedge(A) = R(C_2(A))$ since all vectors in $\wedge^2\mathbb{R}^3$ are decomposable. By a result of [15, Lemma 6], $P_2^\wedge(A)$ is an elliptical disk centered at the origin. To be precise, if $u, v \in \mathbb{R}^3$ are orthogonal, then by direct computation

$$v^T A u = \det \begin{bmatrix} z & -y & x \\ v_1 & v_2 & v_3 \\ u_1 & u_2 & u_3 \end{bmatrix}.$$

Choose the unique $w \in \mathbb{S}_\mathbb{R}^2$ so that $[u, v, w] \in \mathrm{SO}(3)$. Then $(z, -y, x)^T = \alpha u + \beta v + \gamma w$, where $\alpha, \beta, \gamma \in \mathbb{C}$. Then

$$v^T A u = \gamma = (z, -y, x) \cdot w.$$

Since $R_2^\wedge(A) = [P_2^\wedge(A)]^2$, direct computation leads to the desired result. □

Remark 5.2. It would be interesting to know if $P_{2k}^\wedge(A)$ and $R_{2k}^\wedge(A)$ are convex or not for general k and for skew symmetric $A \in \mathbb{C}_{n\times n}$ (even the case $R_2^\wedge(A)$ is unknown).

6. Congruence case

Given $A \in \mathbb{C}_{n\times n}$ and $1 \le k \le n$, if unitary similarity in the formulation (1.1) of $W_k^\wedge(A)$ is replaced by unitary congruence, we have

$$W_k^T(A) := \{\det((U^T A U)[k|k]) : U \in \mathrm{U}(n)\}.$$

It admits circular symmetry, i.e., if $\alpha \in W_k^T(A)$, then $e^{i\theta}\alpha \in W_k^T(A)$ for all $\theta \in \mathbb{R}$. It is because

$$\det((e^{i\theta U} U)^T A(e^{i\theta} U))[k|k]) = e^{i2\theta} \det((U^T A U)[k|k]).$$

Similarly if $B \in \mathbb{C}_{n\times n}$ is skew symmetric, then we define

$$P_{2k}^T(B) := \{\mathrm{Pf}\,((U^T B U)[k|k]) : U \in \mathrm{U}(n)\}.$$

The two sets have the relation $W_{2k}^T(B) = (P_{2k}^T(B))^2$.

Theorem 6.1. (a) Let $A \in \mathbb{C}_{n\times n}$.

(i) If $1 \le k < n$, then $W_k^T(A)$ is a circular disk centered at the origin.

(ii) If $k = n$, then $W_n^T(A)$ is a circle centered at the origin with radius $\det A$.

(b) Let $A \in \mathbb{C}_{n\times n}$ be skew symmetric.

(i) If $2 \le 2k < n$, then $P_{2k}^T(A)$ is a circular disk centered at the origin.

(ii) If $2k = n$, then $P_n^T(A)$ is a circle centered at the origin with radius $\operatorname{Pf} A$.

Proof. (a) The case $k = n$ is trivial. Suppose $1 \le k < n$. Clearly $W_k^T(A)$ is compact. The function $\varphi : \mathrm{U}(n) \to \mathbb{C}$ defined by

$$\varphi(U) := \det((U^T AU)[k|k])$$

is continuous and $\varphi(\mathrm{U}(n)) = W_k^T(A)$. Moreover $\varphi(\alpha U) = \alpha^k \varphi(U)$ and n does not divide k. By [2, Theorem 2], $W_k^T(A)$ is a circular disk centered at the origin.

(b) The case $2k = n$ is trivial. Similarly the function $\psi : \mathrm{U}(n) \to \mathbb{C}$ defined by

$$\psi(U) := \operatorname{Pf}((U^T AU)[2k|2k])$$

is continuous and $\psi(\mathrm{U}(n)) = P_{2k}^T(A)$. Moreover $\psi(\alpha U) = \alpha^k \psi(U)$ and n does not divide k. By [2, Theorem 2], $P_{2k}^T(A)$ is a circular disk centered at the origin. □

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