Generalization of Ky Fan-Amir-Moeź-Horn-Mirsky's Result on the Eigenvalues and Real Singular Values of a Matrix

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Given $A \in \mathbb{C}_{n\times n}$ with e-values $\lambda \in \mathbb{C}^n$
Hermitian part $\frac{A^*+A}{2}$ has e-values $\alpha \in \mathbb{R}^n$.

Ky Fan (1951): $\operatorname{Re}\lambda \prec \alpha$.

Amir-Moéz and Horn (1958), Mirsky (1958), Sherman and Thompson (1972):
the converse is true.

Abbreviation: FAHM for Ky Fan, Amir-Moéz, Horn and Mirsky

Goals:
1. Generalize the results in the context of complex semisimple Lie algebras.
2. Derive inequalities for classical cases $\mathfrak{a}_n, \mathfrak{b}_n, \mathfrak{c}_n, \mathfrak{d}_n$.
3. The real case is also discussed.

Chapter 1. Introduction

Hermitian decomposition of $A \in \mathbb{C}_{n\times n}$:

$$A = \frac{1}{2}(A - A^*) + \frac{1}{2}(A + A^*).$$

The e-values of $A_1 = \frac{1}{2}(A + A^*)$ and $A_2 = \frac{1}{2i}(A - A^*)$ are called real and imaginary s-values.

Majorization: $a, b \in \mathbb{R}^n$, $a \prec b$ if

$$\begin{aligned} \sum_{i=1}^{k} a_{[i]} &\le \sum_{i=1}^{k} b_{[i]}, \quad k = 1, \ldots, n-1, \\ \sum_{i=1}^{n} a_{[i]} &= \sum_{i=1}^{n} b_{[i]}. \end{aligned}$$

Theorem 1.2 (Ky Fan, 1951) Given $A \in \mathbb{C}_{n\times n}$, the real parts of the e-values $\lambda \in \mathbb{C}^n$ of A is majorized by the real s-values $\alpha \in \mathbb{R}^n$ of A, i.e., $\mathrm{Re}\,\lambda \prec \alpha$.

Theorem 1.3 (Amir-Moéz-Horn and Mirsky, 1958) If $\lambda \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}^n$ such that $\mathrm{Re}\,\lambda \prec \alpha$, then there exists $A \in \mathbb{C}_{n\times n}$ such that λ's are the e-values of A and α's are the real s-values of A.

An equivalent form:

Theorem 1.4 (Sherman and Thompson, 1972) If H is a given Hermitian matrix with e-values $\beta \in \mathbb{R}^n$ and if $\alpha \in \mathbb{R}^n$ satisfies $\alpha \prec \beta$, then there exists a skew Hermitian matrix K such that α is the real part of the e-values of $K + H$.

A framework $\longrightarrow$ semisimple Lie algebras

Motivation: A translation of A: $A + \xi I$ for $\xi \in \mathbb{C}$, translates the e-values of A by ξ and the real s-values of A by $\mathrm{Re}\,\xi$. Thus it suffices to consider those $A \in \mathbb{C}_{n\times n}$ such that $\mathrm{tr}\,A = 0$ in FAHM's result.

Let us look at the formulation of FAHM's result:

1. Eigenvalues

(a) Matrix setting:

$$A \underset{\text{similar}}{\overset{\text{unitary}}{\longrightarrow}} B(\text{upper tri}) \underset{\text{entry}}{\overset{\text{diag}}{\longrightarrow}} \text{e-values}$$

(b) Lie algebra:

$$A \overset{\text{AdK}}{\longrightarrow} B(\in \mathfrak{b}) \underset{\rho : \mathfrak{g} \to \mathfrak{h}}{\overset{\text{proj}}{\longrightarrow}} \text{"e-values"} ,$$

where $\mathfrak{b}$ is a Borel subalgebra and $\mathfrak{h}$ is a Cartan subalgebra.

2. Majorization

(a)

$$\alpha \prec \beta \overset{\text{HLP}}{\iff} \alpha \in \operatorname{conv} S_n\beta$$

where conv: convex hull, S_n: full symmetric group.

(b) semisimple Lie algebra

Kostant's result

$$\begin{array}{l} \alpha,\ \beta \in (i\mathfrak{t})_+ \\ \beta - \alpha \in \operatorname{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+ \end{array} \iff \alpha \in \operatorname{conv} W\beta$$

Real singular values $\mathfrak{sl}(n, \mathbb{C})$:

$\mathfrak{k} \oplus i\mathfrak{k}$ Cartan decomposition

Ky Fan's result may be stated as

$$\pi(\mathrm{Ad}K(X+Z)\cap\mathfrak{b})\subset \mathrm{conv}\,WZ, \qquad (1)$$

where $Z\in i\mathfrak{t}$, $X\in\mathfrak{k}$.

AHM's result (in the version of Sherman and Thompson) may be written as

$$\mathrm{conv}\,WZ\subset \cup_{X\in\mathfrak{k}}\pi(\mathrm{Ad}K(X+Z)\cap\mathfrak{b}). \qquad (2)$$

Combining (1) and (2) we have

$$\cup_{X\in\mathfrak{k}}\pi(\mathrm{Ad}K(X+Z)\cap\mathfrak{b}) = \mathrm{conv}\,WZ. \qquad (3)$$

Notice that (3) may be stated as

$$\pi((\mathfrak{k}+\mathrm{Ad}K(Z))\cap\mathfrak{b}) = \mathrm{conv}\,WZ.$$

- Chapter 1: Introduction
- Chapter 2: A proof of FAHM's result
- Chapter 3: Basics of complex Lie algebras
- Chapter 4: Extend FAHM's result to complex semisimple Lie algebras.
- Chapter 5 & 6: Inequalities corresponding to the classical Lie algebras
- Chapter 7: Examine real Lie algebras
- Chapter 8: Inequalities relating the e-values, the real and imaginary s-values for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{R})$.

Chapter 2. A proof of FAHM's result

Theorem 2.9 (FAHM) Let $A \in \mathbb{C}_{n\times n}$ with e-values $\lambda \in \mathbb{C}^n$ and real s-values $\alpha \in \mathbb{R}^n$. Then $\mathrm{Re}\,\lambda \prec \alpha$. Conversely, if $\lambda \in \mathbb{C}^n$, $\alpha \in \mathbb{R}^n$ such that $\mathrm{Re}\,\lambda \prec \alpha$, then there exists $A \in \mathbb{C}_{n\times n}$ with e-values λ's and real s-values α's.

Proof: (Ky Fan) If A has e-values $\lambda_1, \dots, \lambda_n$, then there exists a unitary matrix $U \in \mathrm{U}(n)$ such that

$$Y := UAU^{-1} = \begin{pmatrix} \lambda_1 & * & * \\ & \ddots & * \\ & & \lambda_n \end{pmatrix}$$

Now $A = U^{-1}YU$ and

$$\frac{A + A^*}{2} = U^{-1}(\frac{Y + Y^*}{2})U.$$

Thus A and Y have the same e-values and real s-values. Now $\frac{1}{2}(Y + Y^*)$ is Hermitian and has diagonal entries $\mathrm{Re}\,\lambda$'s and e-values α's. By a result of Schur, $\mathrm{Re}\,\lambda \prec \alpha$.

(AHM) Conversely, suppose $\lambda \in \mathbb{C}^n$ and $\alpha \in \mathbb{R}^n$ such that $\mathrm{Re}\,\lambda \prec \alpha$. By a result of Horn, there is a Hermitian matrix $H = (h_{ij}) \in \mathbb{C}_{n\times n}$ with e-values α's and diagonal entries $\mathrm{Re}\,\lambda$'s. The upper triangular matrix

$$A := \begin{pmatrix} \lambda_1 & 2h_{12} & \dots & 2h_{1n} \\ & \lambda_2 & \dots & 2h_{2n} \\ & & \ddots & \vdots \\ & & & \lambda_n \end{pmatrix} \in \mathbb{C}_{n\times n}$$

has e-values λ's and real s-values α's since $\frac{1}{2}(A + A^*) = H$, of which the e-values are α's.

Key elements:

Schur's triangularization theorem, Schur's result $\longrightarrow$ Ky Fan's result.

Horn's result $\longrightarrow$ AHM's result.

Chapter 3. Preliminaries

$\mathfrak{g}$ = a complex semisimple Lie algebra.

Adjoint representation of G:

$$\mathrm{Ad} : G \to \mathrm{Aut}\,(\mathfrak{g})$$

where $\mathrm{Ad}(g)$ is the derivative of $i_g : G \to G$, $s \mapsto gsg^{-1}$ at the identity.

Adjoint representation of $\mathfrak{g}$:

$$\mathrm{ad}\ : \mathfrak{g} \mapsto \mathrm{End}\,\mathfrak{g}, \quad (\mathrm{ad}\,X)(Y) = [X, Y]$$

Killing form: $\quad B(X, Y) = \mathrm{tr}\,(\mathrm{ad}\,X \circ \mathrm{ad}\,Y)$

Cartan's Theorem: $\mathfrak{g}$ is semisimple if and only if the Killing form of $\mathfrak{g}$ is nondegenerate.

Cartan subalgebra $\mathfrak{h}$: a maximal abelian subalgebra and $\mathrm{ad}_{\mathfrak{g}}H$ is semisimple for all $H \in \mathfrak{h}$, i.e., diagonalizable.

Set

$$\mathfrak{g}^\alpha := \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \;\; \forall H \in \mathfrak{h}\}.$$

Then $\mathfrak{g}$ is a direct sum of the $\mathfrak{g}^\alpha$.

Root system: Δ is the set of all nonzero α such that $\mathfrak{g}^\alpha \neq 0$.

Proposition 3.8 (root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$)

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \dot{\sum}_{\alpha \in \Delta} \mathfrak{g}^\alpha. \tag{4}$$

(a) $\mathfrak{h} = \mathfrak{g}^0$ and $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$,

(b) $B|_{\mathfrak{h}\times\mathfrak{h}}$ is nondegenerate; so there is an vector space isomorphism $\tau : \mathfrak{h}^* \to \mathfrak{h}$ such that $H_\alpha := \tau(\alpha)$, $\alpha \in \mathfrak{h}^*$ satisfies

$$\alpha(H) = B(H, H_\alpha)$$

for all $H \in \mathfrak{h}$,

(c) Δ spans $\mathfrak{h}^*$, the dual space of $\mathfrak{h}$,

Remark 3.9 The space $V := \sum_{\alpha\in\Delta} \mathbb{R}\alpha$ has a real inner product defined by

$$\langle\varphi, \psi\rangle = B(H_\varphi, H_\psi) = \varphi(H_\psi) = \psi(H_\varphi),$$

$\varphi, \psi \in \mathfrak{h}^*$, where H_α is defined by (b). Furthermore, $V|_{\mathfrak{h}_0} = \mathfrak{h}_0^*$, where $\mathfrak{h}_0 = \sum_{\alpha\in\Delta} \mathbb{R}H_\alpha$.

Weyl group: $W = W(\mathfrak{g}, \mathfrak{h})$ is generated by the reflections

$$s_\alpha(\varphi) := \varphi - \frac{2\langle\varphi, \alpha\rangle}{\langle\alpha, \alpha\rangle}\alpha,$$

for all $\varphi \in V$ where $\alpha \in \Delta$, carry Δ onto itself.

Weyl Chambers: Components of V divided by the hyperplanes defined by $\alpha \in \Delta$,

$$\langle\alpha, \xi\rangle = 0.$$

The group W permute the Weyl chambers transitively.

Under the identification W acts on $\mathfrak{h}_0$.

Chapter 4. The complex semisimple case

K: a real compact connected semisimple Lie group.

G: complexification of K

$\mathfrak{k}$, $\mathfrak{g}$ Lie algebras

$\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$.

Fix a maximal torus T of K with Lie algebra denoted by $\mathfrak{t}$.

$\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\alpha},$$

where Δ is the root system of $(\mathfrak{g}, \mathfrak{h})$.

Borel subalgebras: maximal solvable subalgebras

(Standard) Borel subalgebra:

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta^+} \mathfrak{g}^\alpha \tag{5}$$

Adjoint group: Int $\mathfrak{g}$ (=AdG) the group generated by $e^{\text{ad}\,X}$

Theorem 4.1 The Borel subalgebras of a complex semisimple Lie algebra $\mathfrak{g}$ are conjugate under Int $\mathfrak{g}$.

Proposition 4.2 (Djoković and Tam, 2003)

(a) The Borel subalgebras of $\mathfrak{g}$ are all conjugate under AdK.

(b) Let $\mathfrak{b}$ be any Borel subalgebra. Then

$$\text{Ad}K(X) \cap \mathfrak{b} \neq \phi$$

Let θ be the Cartan involution of the (real) Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$.

$\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta(\mathfrak{g}^{\alpha}) = \mathfrak{g}^{-\alpha}$ for all $\alpha \in \Delta$.

$\pi : \mathfrak{g} \to i\mathfrak{t}$, orthogonal projection

Theorem 4.5 (Kostant 1973)

$$\pi(\mathrm{Ad}\, K(Z)) = \mathrm{conv}\, WZ \tag{6}$$

Theorem 4.6 (Extension of FAHM's result) Let $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ be complex semisimple. If $Z \in i\mathfrak{t}$, then

$$\cup_{X \in \mathfrak{k}} \pi(\mathrm{Ad}\, K(X + Z) \cap \mathfrak{b}) = \mathrm{conv}\, WZ, \tag{7}$$

where $\mathfrak{b}$ is given in (5) and W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Equivalently

$$\pi((\mathfrak{k} + \mathrm{Ad}\, K(Z)) \cap \mathfrak{b}) = \mathrm{conv}\, WZ. \tag{8}$$

Proof: We prove $\pi((\mathfrak{k}+\mathrm{Ad}K(Z))\cap\mathfrak{b}) = \mathrm{conv}\, WZ$

'$\subset$':

Let $Y \in \mathfrak{k} + \mathrm{Ad}K(Z)$. Then $Y = X + \mathrm{Ad}k(Z)$ for some $X \in \mathfrak{k}$ and $k \in K$. By Kostant's result (6),

$$\pi(\mathrm{Ad}K(Z)) = \mathrm{conv}\, WZ,$$

and with the fact that $\mathfrak{k} \perp i\mathfrak{t}$ we have

$$\begin{aligned} \pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) &\subset \pi(\mathfrak{k} + \mathrm{Ad}K(Z)) \\ &\subset \mathrm{conv}\, WZ. \end{aligned} \tag{9}$$

'$\supset$':

Conversely, let $\beta \in \mathrm{conv}\, WZ$. By Kostant's result (6) again, there exists $Y \in \mathrm{Ad}K(Z)$ such that $\pi(Y) = \beta$. Then Y can be decomposed as

$$Y = Y_0 + \sum_{\alpha\in\Delta^+} (Y_\alpha + Y_{-\alpha}),$$

$Y_\alpha \in \mathfrak{g}^\alpha$ and $Y_{-\alpha} \in \mathfrak{g}^{-\alpha}$. Since $\theta(Y) = -Y$,

$$\begin{aligned} & -Y_0 + \sum_{\alpha\in\Delta^+} (-Y_\alpha - Y_{-\alpha}) \\ = \; & \theta Y_0 + \sum_{\alpha\in\Delta^+} (\theta Y_\alpha + \theta Y_{-\alpha}). \end{aligned}$$

Since the sums are direct and $\theta(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$, it follows that $Y_0 \in \mathfrak{h} \cap i\mathfrak{k} = i\mathfrak{t}$ and

$$Y = Y_0 + \sum_{\alpha\in\Delta^+}(Y_\alpha - \theta Y_\alpha).$$

Set

$$X := \sum_{\alpha\in\Delta^+}(Y_\alpha + \theta Y_\alpha) \in \mathfrak{k}.$$

Then

$$\begin{aligned} X + Y &= Y_0 + 2\sum_{\alpha\in\Delta^+} Y_\alpha \\ &\in (X + \mathrm{Ad}K(Z)) \cap \mathfrak{b}. \end{aligned}$$

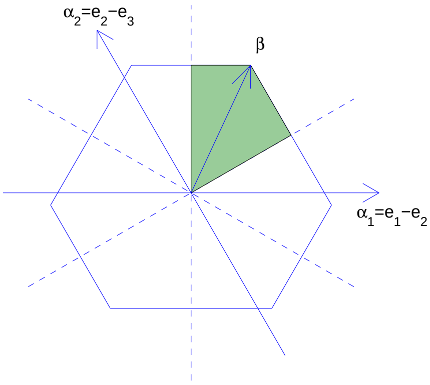
Clearly $\pi(X + Y) = \pi(Y) = \beta$. This proves

$$\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) \supset \mathrm{conv}\, WZ. \qquad (10)$$

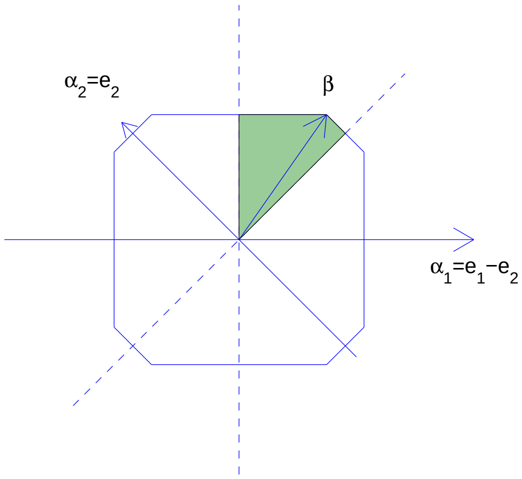
So $\pi((\mathfrak{k} + \mathrm{Ad}K(Z)) \cap \mathfrak{b}) = \mathrm{conv}\, WZ$.

Remark 4.7 When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, the theorem is simply FAHM's result with an appropriate translation.

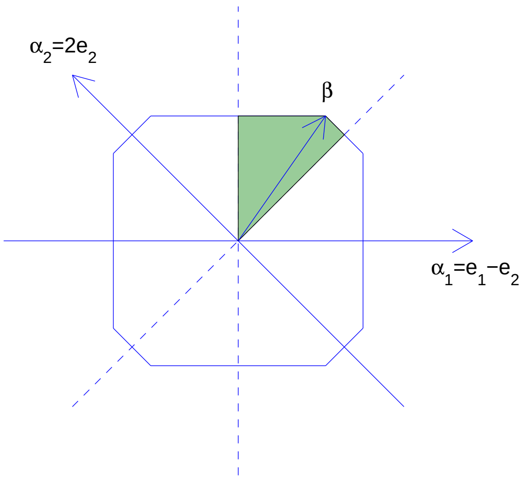
Pictures of the convex hull $\mathrm{conv}\, W\beta$ for $\beta \in i\mathfrak{t}$ for some low dimensional cases.



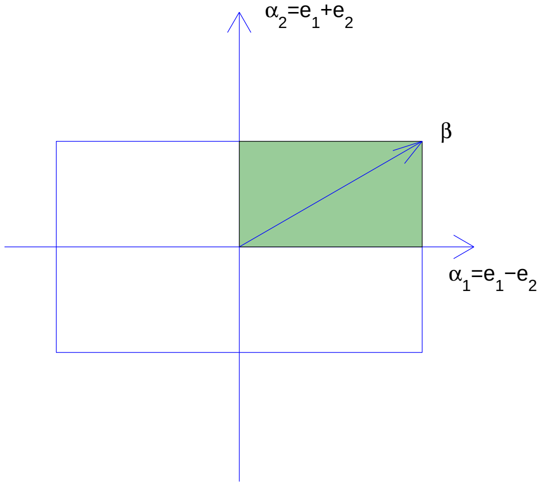
The Convex Hull conv $W\beta$ For $\mathfrak{a}_2$



The Convex Hull conv $W\beta$ For $\mathfrak{b}_2$



The Convex Hull conv $W\beta$ For $\mathfrak{c}_2$



The Convex Hull

conv $W\beta$ For $\mathfrak{d}_2$

Chapter 5. The inequalities for $\mathfrak{a}_n$ and $\mathfrak{c}_n$

$\mathfrak{g}$: a complex semisimple Lie algebra

$\mathfrak{h}$: a Cartan subalgebra

$\mathfrak{b} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta^+} \mathfrak{g}^\alpha$: a Borel subalgebra

$\Pi = \{\alpha_j,\ j = 1, \ldots, n\}$: set of simple roots

$\rho : \mathfrak{g} \to \mathfrak{h}$, $\pi : \mathfrak{g} \to i\mathfrak{t}$ the orthogonal projections

$V = \sum_{i=1}^n \mathbb{R}\alpha_i \quad \mathfrak{h}_0 = i\mathfrak{t}$

Fundamental dominant weights (FDW) are the basis dual to $\frac{2\alpha_i}{(\alpha_i,\alpha_i)}$, $i = 1, \ldots, n$:

$$\lambda_1, \ldots, \lambda_n$$

Fundamental Weyl chamber (FWC):

$$(i\mathfrak{t})_+ = \{H \in i\mathfrak{t} : \alpha_j(H) \geq 0,\ j = 1, \ldots, n\}.$$

Proposition 5.1 The FWC $(i\mathfrak{t})_+$ is the cone

$$C = \{\sum_{j=1}^{n} a_j\lambda_j : a_j \geq 0, j = 1, \ldots, n\}$$

generated by $\lambda_j,\ j = 1, \ldots, n$.

The dual cone $\mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+ \subset i\mathfrak{t}$ of $(i\mathfrak{t})_+$

$$\begin{aligned} & \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+ \\ := \ & \{X \in i\mathfrak{t} : (X, Y) \geq 0, \forall\, Y \in (i\mathfrak{t})_+\} \\ = \ & \{X \in i\mathfrak{t} : (X, \lambda_j) \geq 0,\ j = 1, \ldots, n\}. \end{aligned}$$

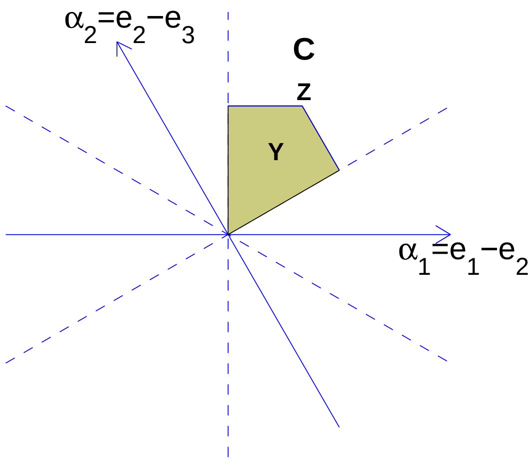
Lemma 5.2 (Kostant 1973)

(a) Let $Z \in (i\mathfrak{t})_+$. Then for all $w \in W$,

$$Z - wZ \in \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+.$$

(b) Let $Y, Z \in (i\mathfrak{t})_+$. Then

$$Y \in \mathrm{conv}\, WZ \iff Z - Y \in \mathrm{dual}_{i\mathfrak{t}}(i\mathfrak{t})_+.$$



$$Z - Y \in C := \text{dual}\,_{i\mathfrak{t}}(i\mathfrak{t})_+ \text{ for } \mathfrak{sl}(3,\mathbb{C})$$

Weak majorization: $a, b \in \mathbb{R}^n$. $a \prec_w b$ if

$$\sum_{i=1}^{k} a_{[i]} \le \sum_{i=1}^{k} b_{[i]}, \qquad k = 1, \ldots, n.$$

In $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C})$,

(a) $\rho : \mathfrak{b} \to \mathfrak{h}$: e-values

(b) $\pi : \mathfrak{b} \to i\mathfrak{t}$: real part of e-values

Inequalities

(a) If $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, then FAHM's result.

(b) If $\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$, then Proposition 5.7: The n largest nonnegative real parts of the e-values of $A \in \mathfrak{sp}(n, \mathbb{C})$ are weakly majorized by the n largest nonnegative real singular values of A. Conversely given α, $\beta \in \mathbb{R}^n$ with positive entries, if $\alpha \prec_w \beta$, then there exists $A \in \mathfrak{sp}(n, \mathbb{C})$ such that $\pm\alpha_1, \ldots, \pm\alpha_n$ are the real parts of the e-values of A and $\pm\beta_1, \ldots, \pm\beta_n$ are the real singular values of A.

Chapter 6. The inequalities for $\mathfrak{b}_n$ and $\mathfrak{d}_n$

Problem : In Chapter 3, models $\mathfrak{g} = \mathfrak{so}(2n+1,\mathbb{C})$ and $\mathfrak{g} = \mathfrak{so}(2n,\mathbb{C})$ for $\mathfrak{b}_n$ and $\mathfrak{d}_n$, respectively. It is not easy to see

$$\rho : \mathfrak{b} \to \mathfrak{h}, \quad \pi : \mathfrak{b} \to i\mathfrak{t}$$

amount taking the e-values and the real parts of the e-values, respectively.
A fix: switch to another model $\tilde{\mathfrak{g}}$.

Lemma 6.3 and Proposition 6.10 For any X in $\mathfrak{so}(2n+1,\mathbb{C})$ or $\mathfrak{so}(2n,\mathbb{C})$,
$\rho(\text{Ad}(K)X \cap \mathfrak{b})$: e-values of X

$\pi(\text{Ad}(K)X \cap \mathfrak{b})$: real part e-values of X

Proposition 6.5 Let A be in $\mathfrak{so}(2n+1,\mathbb{C})$ or $\mathfrak{so}(2n,\mathbb{C})$. Let $\pm\beta_1,\ldots,\pm\beta_n$ be the real singular values of A. Let

(a)

$$\begin{pmatrix} 0 & i\alpha_1 \\ -i\alpha_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & i\alpha_n \\ -i\alpha_n & 0 \end{pmatrix} \oplus (0)$$

$$\in \pi(\mathrm{Ad}K(A) \cap \mathfrak{b}),$$

if $X \in \mathfrak{so}(2n+1,\mathbb{C})$,

(b) or

$$\begin{pmatrix} 0 & i\alpha_1 \\ -i\alpha_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & i\alpha_n \\ -i\alpha_n & 0 \end{pmatrix}$$

$$\in \pi(\mathrm{Ad}K(A) \cap \mathfrak{b}),$$

if $X \in \mathfrak{so}(2n,\mathbb{C})$

Thus $\pm\alpha_1,\ldots,\pm\alpha_n$ are the real parts of the eigenvalues of A. Then

(a)

$$\sum_{j=1}^{k} |\alpha|_{[j]} \leq \sum_{j=1}^{k} |\beta|_{[j]}, \quad k = 1, \ldots, n,$$

(Weak majorization) if $X \in \mathfrak{so}(2n+1, \mathbb{C})$

(b)

$$\begin{aligned}
\sum_{j=1}^{k} |\alpha|_{[j]} &\leq \sum_{j=1}^{k} |\beta|_{[j]}, \; k = 1, \ldots, n - \\
\sum_{j=1}^{n-1} |\alpha|_{[j]} + \delta_1 |\alpha|_{[n]} &\leq \sum_{j=1}^{n-1} |\beta|_{[j]} + \delta_2 |\beta|_{[n]}, \\
\sum_{j=1}^{n-1} |\alpha|_{[j]} - \delta_1 |\alpha|_{[n]} &\leq \sum_{j=1}^{n-1} |\beta|_{[j]} - \delta_2 |\beta|_{[n]},
\end{aligned}$$

where

$$\begin{aligned}
\delta_1 &= \text{sign}(\alpha_1 \cdots \alpha_n), \\
\delta_2 &= \text{sign}\,[(-i)^n \text{Pf}\,(\frac{1}{2}(A + A^*))]
\end{aligned}$$

if $X \in \mathfrak{so}(2n, \mathbb{C})$.

Conversely, if above inequalities hold for some real n-tuples α and β, then we can find A in $\mathfrak{so}(2n+1,\mathbb{C})$ and $\mathfrak{so}(2n,\mathbb{C})$, respectively, satisfies

(a) $\pm\alpha$, 0 are the real part of the eigenvalues of A, $\pm\beta$, 0 are the real singular values of A,

(b) $\pm\alpha$ are the real part of the eigenvalues of A,
$\pm\beta$ are the real singular values of A, and

$$\text{sign}\,[(-i)^n \text{Pf}\,(\frac{1}{2}(A+A^*))] = \text{sign}\,(\beta_1 \cdots \beta_n).$$

Remark: The difference between the cases for $2n+1$ and $2n$ comes from the different actions of their Weyl groups.

(a) $\mathfrak{so}(2n+1, \mathbb{C})$, the Weyl group acts on $i\mathfrak{t}$ by:

$$(h_1, \ldots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \ldots, \pm h_{\sigma(n)})^T, \quad \sigma \in S_n.$$

(b) $\mathfrak{so}(2n, \mathbb{C})$, the Weyl group acts on $i\mathfrak{t}$ by:

$$(h_1, \ldots, h_n)^T \mapsto (\pm h_{\sigma(1)}, \ldots, \pm h_{\sigma(n)})^T, \quad \sigma \in S_n,$$

where the number of negative signs is even.

Chapter 7. The real semisimple case

$\mathfrak{g}$ = a real semisimple Lie algebra

$\mathfrak{h}$ = a Cartan subalgebra of $\mathfrak{g}$

$B(\cdot,\cdot)$ the Killing form on $\mathfrak{g}$

Example 7.1 There are non-conjugate Cartan subalgebras: $\mathfrak{sl}(2,\mathbb{R})$

$$\mathfrak{a} = \mathbb{R}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathfrak{b} = \mathbb{R}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Cartan decomposition of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$:

(a) $\theta : X + Y \to X - Y \ (X \in \mathfrak{k}, Y \in \mathfrak{p})$

(b) $B_\theta(X,Y) = -B(X,\theta Y)$ inner product on $\mathfrak{g}$

$\mathfrak{a}_\mathfrak{p}$ = maximal abelian subspace in $\mathfrak{p}$

A Cartan subalgebra $\mathfrak{a}$ containing $\mathfrak{a}_\mathfrak{p}$ is of the form

$$\mathfrak{a} = \mathfrak{a}_\mathfrak{k} \oplus \mathfrak{a}_\mathfrak{p}, \quad \mathfrak{a}_\mathfrak{k} = \mathfrak{a} \cap \mathfrak{k}, \quad \mathfrak{a}_\mathfrak{p} = \mathfrak{a} \cap \mathfrak{p}.$$

Since $(\mathrm{ad}\, H)^* = \mathrm{ad}\, H$, for all $H \in \mathfrak{a}_\mathfrak{p}$ and $\mathrm{ad}\, \mathfrak{a}_\mathfrak{p}$ is abelian,

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \dot{\sum}_{\alpha \in \Sigma^+} (\mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}), \qquad (11)$$

is the restricted root space decomposition of $\mathfrak{g}$ relative to $\mathfrak{a}_\mathfrak{p}$, Σ^+ the set of restricted positive roots.

$N(\mathfrak{a}_\mathfrak{p})$ = normalizer of $\mathfrak{a}_\mathfrak{p}$ in K
$Z(\mathfrak{a}_\mathfrak{p})$ = centralizer of $\mathfrak{a}_\mathfrak{p}$ in K

Weyl group W: $N(\mathfrak{a}_\mathfrak{p})/Z(\mathfrak{a}_\mathfrak{p})$

$\pi_{\mathfrak{a}_\mathfrak{p}} : \mathfrak{g} \to \mathfrak{a}_\mathfrak{p}$ orthogonal projection

Theorem 7.2 (Kostant 1973) For $Z \in \mathfrak{a}_\mathfrak{p}$,

$$\pi_{\mathfrak{a}_\mathfrak{p}}(\mathrm{Ad} K(Z)) = \mathrm{conv}\, WZ. \qquad (12)$$

Theorem 7.4 Let $\mathfrak{a} \supset \mathfrak{a}_{\mathfrak{p}}$ be a Cartan subalgebra, $\mathfrak{b} = \mathfrak{a} \dot{+} \dot{\sum}_{\alpha \in \Sigma^+} \mathfrak{g}^{\alpha}$. Then for each $\beta \in \mathfrak{a}_{\mathfrak{p}}$,

$$\pi((\mathfrak{k} + \mathrm{Ad} K(\beta)) \cap \mathfrak{b}) = \mathrm{conv}\, W\beta.$$

Proof: Similar to Theorem 4.6.

Remark 7.5 Theorem 7.4 is a AHM's type result for the real semisimple Lie algebras.

Proposition 7.6 For any $X \in \mathfrak{sl}(n, \mathbb{R})$, there exists $k \in \mathrm{SO}(n)$ such that kXk^{-1} is of block upper triangular form where the (main diagonal) blocks are either 1×1 or 2×2:

$$kXk^{-1} = \begin{pmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ & & & A_s \end{pmatrix},$$

with zero trace, where $A_k = \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix}$, or $A_k = (c_k)$, $a_k, b_k, c_k \in \mathbb{R}$, $k = 1, \ldots, s$.

Remark Unlike the complex case,

(1) given $X \in \mathfrak{g}$, the adjoint orbit $\mathrm{Ad}K(X)$ may not intersect a specific maximal solvable subalgebra $\mathfrak{b}$.

(2) two maximal solvable subalgebras may not be conjugate.

A Cartan subalgebra $\mathfrak{c}$ is called a standard Cartan subalgebra if $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$, where

$$\mathfrak{a}_{\mathfrak{k}} \subset \mathfrak{c}_{\mathfrak{k}} := \mathfrak{c} \cap \mathfrak{k}, \qquad \mathfrak{c}_{\mathfrak{p}} := \mathfrak{c} \cap \mathfrak{p} \subset \mathfrak{a}_{\mathfrak{p}}.$$

Proposition 7.14 $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$ standard Cartan subalgebra, $\pi_{\mathfrak{c}_{\mathfrak{p}}} : \mathfrak{g} \to \mathfrak{c}_{\mathfrak{p}}$ orthogonal projection. Then

$$\mathrm{conv}\, W\beta \cap \mathfrak{c}_{\mathfrak{p}} = \pi_{\mathfrak{c}_{\mathfrak{p}}}(\mathrm{Ad}K(\beta)) = \pi_{\mathfrak{c}_{\mathfrak{p}}}(\mathrm{conv}\, W\beta).$$

Let $\mathfrak{s}$ be a standard maximal solvable subalgebra containing the standard Cartan subalgebra

$\mathfrak{c}$. Since $\mathfrak{c}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p} \subset \mathfrak{p}$, $\mathfrak{c}_\mathfrak{p} \perp \mathfrak{k}$ so that

$$\begin{aligned} \pi_{\mathfrak{c}_\mathfrak{p}}((\mathfrak{k} + \mathrm{Ad}K(\beta)) \cap \mathfrak{s}) &\subset \pi_{\mathfrak{c}_\mathfrak{p}}(\mathfrak{k} + \mathrm{Ad}K(\beta)) \\ &= \pi_{\mathfrak{c}_\mathfrak{p}}(\mathrm{Ad}K(\beta)) \\ &= \mathrm{conv}\, W\beta \cap \mathfrak{c}_\mathfrak{p}. \end{aligned}$$

Question: Does set equality hold?

Answer: No. Not even for some normal real Lie algebras.

Example 7.15 $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$.
$\mathfrak{k}$ real skew symmetric matrices
$\mathfrak{p}$ real symmetric matrices
$\mathfrak{a}_\mathfrak{p} = \text{diag}\,(a, b, -a-b)$
$\mathfrak{c}_\mathfrak{p} = \text{diag}\,(a, a, -2a)$.
Let $\beta = \text{diag}\,(1, 1, -2)$.
Pick

$$H = (-\frac{1}{2}, -\frac{1}{2}, 1) \in \text{conv}\, W\beta \cap \mathfrak{c}_\mathfrak{p}$$

$$H \notin \pi_{\mathfrak{c}_\mathfrak{p}}\{(\mathfrak{k} + \text{Ad}\,K(\beta)) \cap \mathfrak{s}_i\},$$

where $i = 1, 2$.

Question: Given $X \in \mathfrak{g}$. Is $\text{Ad}\,K(X) \cap \mathfrak{s} \neq \phi$ for some standard maximal solvable subalgebra $\mathfrak{s}$?

Partial answer: Yes for compact $\mathfrak{g}$.

Proposition 7.12 $\text{Ad}\,K(X) \cap \mathfrak{s} \neq \phi$ if $\mathfrak{g}$ is compact.

General answer: No, in general.

Example 7.13

$$\mathrm{SU}(1,1)=\left\{\begin{pmatrix}\alpha & \beta\\ \overline{\beta} & \overline{\alpha}\end{pmatrix} : |\alpha|^2-|\beta|^2=1\right\},$$

whose Lie algebra is a real form of $\mathfrak{sl}(2,\mathbb{C})$:

$$\begin{aligned}
\mathfrak{su}_{1,1} &= \left\{\begin{pmatrix} ia & c\\ \bar{c} & -ia\end{pmatrix} : a\in\mathbb{R}, c\in\mathbb{C}\right\},\\
K &= \left\{\mathrm{diag}\,(e^{i\theta}, e^{-i\theta}) : \theta\in\mathbb{R}\right\},\\
\mathfrak{k} &= \left\{\begin{pmatrix} ia & 0\\ 0 & -ia\end{pmatrix} : a\in\mathbb{R}\right\},\\
\mathfrak{p} &= \left\{\begin{pmatrix} 0 & c\\ \bar{c} & 0\end{pmatrix} : c\in\mathbb{C}\right\},\\
\mathfrak{a}_{\mathfrak{p}} &= \left\{\begin{pmatrix} 0 & b\\ b & 0\end{pmatrix} : b\in\mathbb{R}\right\}.
\end{aligned}$$

Two non-conjugate standard maximal solvable subalgebras: $\mathfrak{s}_1=\mathfrak{k}$ and

$$\mathfrak{s}_2=\mathfrak{a}_{\mathfrak{p}}+\mathfrak{g}^{\alpha}(H)=\mathbb{R}\begin{pmatrix}-ia & ia+b\\ -ia+b & ia\end{pmatrix}$$

But for $j=1,2$,

$$\mathrm{Ad}K\left(\begin{pmatrix}-i & \epsilon\\ \epsilon & i\end{pmatrix}\right)\cap\mathfrak{s}_j=\phi,\quad \text{if } \epsilon<1.$$

Theorem 7.7 (Sugiura 1959) $\mathfrak{g}$ a real semisimple Lie algebra

(1) Every Cartan subalgebra of $\mathfrak{g}$ is conjugate to a standard Cartan subalgebra via $\mathrm{Int}\,(\mathfrak{g})$.

(2) Two standard Cartan subalgebras are conjugate via $\mathrm{Int}\,(\mathfrak{g})$ if and only if their vector parts are conjugate under the Weyl group W of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$.

$\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$ standard Cartan subalgebra,
$\mathfrak{c}_{\mathfrak{p}} \subset \mathfrak{a}_{\mathfrak{p}}$,
$\mathfrak{g}$ has a root space decomposition with respect to $\mathfrak{c}_{\mathfrak{p}}$:

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \sum_{\alpha \in R} \mathfrak{g}^{\alpha},$$

$H \in \mathfrak{c}_{\mathfrak{p}}$ is called $\mathfrak{c}_{\mathfrak{p}}$-singular if there exists $\alpha \in R$ such that $\alpha(H) = 0$, otherwise H is called $\mathfrak{c}_{\mathfrak{p}}$-general.

A connected component in the set of $\mathfrak{c}_{\mathfrak{p}}$-general elements of $\mathfrak{c}_{\mathfrak{p}}$ is called a $\mathfrak{c}_{\mathfrak{p}}$-chamber.

Let C be a $\mathfrak{c}_{\mathfrak{p}}$-chamber in $\mathfrak{c}_{\mathfrak{p}}$. Then $\mathfrak{g}^{\alpha}(H)$ is independent of the choice of H in C and thus may be written as $\mathfrak{g}^{\alpha}(C)$.

Theorem 7.9 (Mostow 1961)

(1) Any maximal solvable subalgebra contains a Cartan subalgebra, hence is conjugate to a standard maximal solvable subalgebra.

(2) Any maximal solvable subalgebra containing a standard Cartan subalgebra $\mathfrak{c} = \mathfrak{c}_{\mathfrak{k}} \dot{+} \mathfrak{c}_{\mathfrak{p}}$ is of the form $\mathfrak{c} + \mathfrak{g}^{+}(C)$ for some $\mathfrak{c}_{\mathfrak{p}}$-chamber C.

standard Cartan subalgebra $\longrightarrow$ non-conjugate $\mathfrak{c}_{\mathfrak{p}} \longrightarrow$ maximal solvable subalgebra

Example 7.11 $\mathfrak{g} = \mathfrak{sl}(3, \mathbb{R})$. Non-conjugate Cartan subalgebras:

$$\mathfrak{c}_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$\mathfrak{c}_2 = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

Non-conjugate maximal solvable subalgebras:

$$\begin{aligned} \mathfrak{s}_1 &:= \left\{ \begin{pmatrix} a & c & e \\ 0 & b & d \\ 0 & 0 & -a-b \end{pmatrix} \right\}, \\ \mathfrak{s}_2 &:= \left\{ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & -2a \end{pmatrix} \right\}, \\ \mathfrak{s}_3 &:= \left\{ \begin{pmatrix} -2a & c & d \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} \right\}. \end{aligned}$$

Chapter 8 The e-values and the real and imaginary s-values for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{R})$

Question: What is the necessary and sufficient condition on $\lambda \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{R}^n$ so that a matrix $A \in \mathbb{C}_{n\times n}$ exists with e-values λ's, real s-values α's and imaginary s-values β's?

Proposition 8.1 (complex case) Let $\alpha, \beta \in \mathbb{R}$ and $a + ib \in \mathbb{C}$. Then there exists $A \in \mathfrak{sl}(2,\mathbb{C})$ whose e-values, real s-values, and imaginary s-values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively, if and only if
(1) $(-a, a) \prec (-\alpha, \alpha)$, $(-b, b) \prec (-\beta, \beta)$, and
(2) $\beta^2 - b^2 = \alpha^2 - a^2$.

Proposition 8.2 (real case) Let $\alpha, \beta \in \mathbb{R}$ and $a + ib \in \mathbb{C}$. Then there exists $A \in \mathfrak{sl}(2,\mathbb{R})$ whose e-values, real s-values, and imaginary s-values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively, if and only if
(1) $b = 0$, $(-a, a) \prec (-\alpha, \alpha)$, and $\beta^2 = \alpha^2 - a^2$, or
(2) $a = \alpha = 0$, $b = \pm\beta$.