Operators and Matrices

[0268] First Galley Proofs

MULTIPLICITIES, BOUNDARY POINTS, AND JOINT NUMERICAL RANGES

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Abstract. The multiplicity of a point in the joint numerical range $W(A_1,A_2,A_3) \subseteq \mathbb{R}^3$ is studied for $n\times n$ Hermitian matrices A_1,A_2,A_3. The relative interior points of $W(A_1,A_2,A_3)$ have multiplicity greater than or equal to $n-2$. The lower bound $n-2$ is best possible. Extreme points and sharp points are studied. Similar study is given to the convex set $V(A) := \{x^TAx : x\in \mathbb{R}^n, x^Tx = 1\} \subseteq \mathbb{C}$, where $A\in \mathbb{C}_{n\times n}$ is symmetric. Examples are given.

1. Introduction

Let $\mathbb{C}_{n\times n}$ be the set of $n\times n$ complex matrices. The classical numerical rang of $A\in \mathbb{C}_{n\times n}$ is

$$W(A) := \{x^*Ax : x\in \mathbb{C}^n, x^*x = 1\}.$$

It is the image of the unit sphere

$$\mathbb{S}^{n-1} = \{x\in \mathbb{C}^n : x^*x = 1\}$$

under the quadratic map $x\mapsto x^*Ax$. Toeplitz-Hausdorff theorem asserts that $W(A)$ is a compact convex set [9]. When $n=2$, $W(A)$ is an elliptical disk (possibly degenerate) [9], known as the elliptical range theorem.

A point $\xi \in W(A)$ is called an extreme point if ξ is not in any open line segment that is contained in $W(A)$. A point $\xi\in W(A)$ is a sharp point if ξ is the intersection point of two distinct supporting lines of $W(A)$ [9, p. 50]. We have the following inclusions for $W(A)$, which are proper in general:

$$\{\text{sharp points}\} \subseteq \{\text{extreme points}\} \subseteq \{\text{boundary points}\}. \tag{1.1}$$

Donoghue [7] showed that sharp points of $W(A)$ are eigenvalues of A. Indeed the following is a characterization of the sharp points.

THEOREM 1.1. ([9, p. 50–51]) *Let $A\in \mathbb{C}_{n\times n}$ and $\xi\in W(A)$. Then ξ is a sharp point if and only if A is unitarily similar to $\xi I\oplus B$ with $\xi\notin W(B)$.*

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Given $\xi \in W(A)$, Embry [8] introduced

$$M_\xi = M_\xi(A) := \{x \in \mathbb{C}^n : x^*Ax = \xi x^*x\}.$$

In general M_ξ is not a subspace but it is homogeneous. Thus the span of M_ξ, denoted by $\langle M_\xi \rangle$, satisfies $\langle M_\xi \rangle = M_\xi + M_\xi := \{x+y : x,y \in M_\xi\}$. Stampfli [14, Lemma 2] showed that M_ξ is a subspace of $\mathbb{C}^n$ if ξ is an extreme point. Embry [8] established the converse and some related results.

THEOREM 1.2. (Embry) *Let $A \in \mathbb{C}_{n\times n}$ and $\xi \in W(A)$. Then*

1. *ξ is an extreme point if and only if M_ξ is a subspace of $\mathbb{C}^n$.*
2. *if ξ is a non-extreme boundary point, then*

$$\langle M_\xi \rangle = \cup_{w\in L\cap W(A)} M_w,$$

where L is the supporting line of $W(A)$, passing through ξ. In this case $\langle M_\xi \rangle = \mathbb{C}^n$ if and only if $W(A) \subseteq L$.

3. *if $W(A)$ is nondegenerate, then ξ is an interior point if and only if $\langle M_\xi \rangle = \mathbb{C}^n$.*

We remark that the results of Dongonhue, Stampfli, and Embry are for any bounded linear operator on a complex Hilbert space.

Now consider for each $\xi \in W(A)$

$$w_A(\xi) := \dim\langle M_\xi \rangle,$$

i.e., $w_A(\xi)$ is the maximal number of linearly independent vectors $x \in \mathbb{S}^{n-1}$ such that $x^*Ax = \xi$. We call $w_A(\xi)$ the *multiplicity* of ξ. It is well known that $W(A)$ is a line segment $[\alpha,\beta]$ if and only if A is essentially Hermitian, in which case $w_A(\xi) = n$ for any $\xi \in (\alpha,\beta)$. With this fact, one can deduce from Theorem 1.2 that $w_A(\xi) = n$ for any relative interior point $\xi \in W(A)$, thus provides an affirmative answer to a question of Uhlig in [15, p. 18].

We will study multiplicities of relative interior points and some characterizations of extreme points and sharp points of two variations of the classical numerical range.

2. Joint numerical range of three Hermitian matrices

Let H_n be the set of $n\times n$ Hermitian matrices. Let $A = A_1 + iA_2$ be the Hermitian decomposition of $A \in \mathbb{C}_{n\times n}$, where $A_1, A_2 \in H_n$. Since

$$x^*Ax = x^*A_1x + ix^*A_2x$$

and $x^*A_1x, x^*A_2x \in \mathbb{R}$, one may identify $W(A)$ as the set

$$W(A_1,A_2) := \{(x^*A_1x, x^*A_2x) : x \in \mathbb{C}^n, x^*x = 1\} \subseteq \mathbb{R}^2.$$

Given $A_1,\dots,A_k \in H_n$, Au-Yeung and Poon [1] and other authors, for example, Binding and Li [4], Au-Yeung and Tsing [3], Li and Poon [10], considered the following generalization of $W(A)$:

$$W(A_1,\dots,A_k) := \{(x^*A_1x,\dots,x^*A_kx) \; : \; x \in \mathbb{C}^n, x^*x = 1\},$$

which is a joint numerical range of $A_1,\dots,A_k$. We remark that $W(A)$ can be viewed as $W(A_1,A_2,0)$. Au-Yeung and Tsing [2] proved that $W(A_1,A_2,A_3)$ is convex when $n \geqslant 3$. Moreover $W(A_1,A_2,A_3)$ is an ellipsoid (possibly degenerate) when $n = 2$ (see [6] for a conceptual reason).

DEFINITION 2.1. For $\xi \in W(A_1,A_2,A_3)$, where $A_1,A_2,A_3 \in H_n$, the multiplicity, denoted by $w_{A_1,A_2,A_3}(\xi)$, is the maximal number of linearly independent vectors $x \in \mathbb{S}^{n-1}$ such that $(x^*A_1x, x^*A_2x, x^*A_3x) = \xi$.

To simplify notation, given $A_1,A_2,A_3 \in H_n$, we let $f : \mathbb{S}^{n-1} \to \mathbb{R}^3$ be the map defined by

$$f(x) := (x^*A_1x, x^*A_2x, x^*A_3x), \quad x \in \mathbb{S}^{n-1}.$$

We denote by $\mathrm{Int}_R S$ the relative interior of $S \subset \mathbb{R}^k$.

THEOREM 2.2. *Let $A_1,A_2,A_3 \in H_n$ and $\xi \in W(A_1,A_2,A_3)$.*

1. *When $n = 2$,*
   (a) *if $W(A_1,A_2,A_3)$ is a nondegenerate ellipsoid, then $w_{A_1,A_2,A_3}(\xi) = 1$.*
   (b) *if $W(A_1,A_2,A_3)$ is a nondegenerate elliptical disk or a nondegenerate line segment, then*
   (i) $w_{A_1,A_2,A_3}(\xi) = 2$ if $\xi \in \mathrm{Int}_R W(A_1,A_2,A_3)$,
   (ii) $w_{A_1,A_2,A_3}(\xi) = 1$ if $\xi \notin \mathrm{Int}_R W(A_1,A_2,A_3)$.
   (c) *if $W(A_1,A_2,A_3) = \{\xi\}$ (in this case A_1,A_2,A_3 are scalar multiples of identity), then $w_{A_1,A_2,A_3}(\xi) = 2$.*
2. *When $n \geqslant 3$, if $\xi \in \mathrm{Int}_R W(A_1,A_2,A_3)$, then*

$$w_{A_1,A_2,A_3}(\xi) \geqslant n-2$$

and may not be a constant. The lower bound $n-2$ is best possible.

Proof. (1) Suppose $n = 2$. Notice that (c) is trivial.

(a) Suppose that $W(A_1,A_2,A_3)$ is a nondegenerate ellipsoid and $\xi \in W(A_1,A_2,A_3)$. Then orthogonally project $W(A_1,A_2,A_3)$ onto the hyperplane $P \subseteq \mathbb{R}^3$ with orthonormal basis $\{p := (p_1,p_2,p_3), q := (q_1,q_2,q_3)\} \subseteq \mathbb{R}^3$ so that the image is a nondegenerate elliptical disk E and the projection ξ' of ξ is on the (relative) boundary of E. With respect to the basis $\{p,q\}$,

$$\begin{aligned} E &= \{(p^T(x^*A_1x, x^*A_2x, x^*A_3x), q^T(x^*A_1x, x^*A_2x, x^*A_3x)) : x \in S^1\} \\ &= \{(x^*(p_1A_1 + p_2A_2 + p_3A_3)x, x^*(q_1A_1 + q_2A_2 + q_3A_3)x : x \in S^1\} \end{aligned}$$

which is naturally identified as $W(A)$, where $A = (p_1A_1 + p_2A_2 + p_3A_3) + i(q_1A_1 + q_2A_2 + q_3A_3)$. By Theorem 1.2, $w_A(\xi') = 1$ and thus $w_{A_1,A_2,A_3}(\xi) = 1$.

The proof of (b) is similar to that of (a).

(2) The statement is trivial for n=3. Suppose that $n \geqslant 4$ and let $\xi \in \mathrm{Int}_R W(A_1,A_2,A_3)$. Suppose on the contrary that $w_{A_1,A_2,A_3}(\xi) = k < n-2$. Let $\{x_1,\ldots,x_k\} \subseteq \mathbb{S}^{n-1}$ be a (maximal) linearly independent set such that $\xi = f(x_i)$, $i = 1,\ldots,k$. Since $k < n-2$, there is $u \in \mathbb{S}^{n-1}$ such that $x_1,\ldots,x_k,u$ are linearly independent and thus $\xi \neq f(u)$. Because $\xi \in \mathrm{Int}_R W(A_1,A_2,A_3)$ and $W(A_1,A_2,A_3)$ is convex in $\mathbb{R}^3$ [2], there is $v \in \mathbb{S}^{n-1}$ such that

(a) $f(v) \neq \xi$ and $f(v) \neq f(u)$,

(b) the line segment $L := [f(u), f(v)] \subseteq W(A_1,A_2,A_3)$, and

(c) $\xi \in L$.

In other words, $\xi \in (f(u), f(v)) \subseteq W(A_1,A_2,A_3)$, where $(f(u), f(v))$ denotes the open line segment. Since $f(u) \neq f(v)$, u,v are linearly independent. Since $k < n-2$, there would exist a unit vector $w \notin \langle u,v,x_1,\ldots,x_k\rangle$. Let $\hat{A}_i$, $i = 1,2,3$, denote the compression of A_i onto the 3-dimensional subspace $\langle u,v,w\rangle$. So $\xi \in W(\hat{A}_1,\hat{A}_2,\hat{A}_3)$ and hence there would exist a unit vector $y \in \langle u,v,w\rangle$ and $f(y) = \xi$. Write $y = \alpha u + \beta v + \gamma w$ for $\alpha,\beta,\gamma \in \mathbb{C}$. Notice that α and γ cannot be both zero, otherwise $f(y) = f(v) \neq \xi$. Then $y,x_1,\ldots,x_k$ would be linearly independent, a contradiction.

The following example shows that the bound $n-2$ is best possible. $\square$

EXAMPLE 2.3. Let $n \geqslant 3$ and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus 0_{n-2},\ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2},\ A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \oplus 0_{n-2}.$$

It is known that $W(A_1,A_2,A_3) \subseteq \mathbb{R}^3$ is the convex hull of $W(B_1,B_2,B_3)$ and the origin, where $B_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Since $W(B_1,B_2,B_3)$ is the unit sphere [1, 10], $W(A_1,A_2,A_3)$ is the unit ball in $\mathbb{R}^3$ and it is not hard to deduce (a) and (b), and (c) can be obtained by direct computation:

(a) $w_{A_1,A_2,A_3}(0) = n-2$,

(b) $w_{A_1,A_2,A_3}(\xi) = 1$ for $|\xi| = 1$,

(c) $w_{A_1,A_2,A_3}(\xi) = n-1$ for $0 < |\xi| < 1$.

DEFINITION 2.4. Let $A_1,A_2,A_3 \in H_n$. A point $\xi \in W(A_1,A_2,A_3)$ is called

1. an extreme point if ξ is not in an open line segment that is contained in $W(A_1,A_2,A_3)$.
2. a sharp point if ξ is the intersection point of three distinct supporting planes of $W(A_1,A_2,A_3)$.

Evidently an extreme point $\xi \in W(A_1,A_2,A_3)$ is a boundary point. However, a boundary point is not necessarily an extreme point. For example, $W(B_1,B_2,B_3) = \text{conv}\{(2,0,0),W(A_1,A_2,A_3)\}$ if $B_1 = A_1 \oplus 2$ and $B_2 = A_2 \oplus 0$ and $B_3 = A_3 \oplus 0$, where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ A_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}.$$

An extreme point may not be a sharp point (see the boundary points in Example 2.3).

Binding and Li [4, Definition 2.4] introduced conical point of any subset $\Sigma \subseteq \mathbb{R}^k$. Since $W(A_1,A_2,A_3)$ is convex when $n \geqslant 3$, sharp points are conical points and the following result is an analog to Theorem 1.1 and can be deduced from [4, Proposition 2.5 or Theorem 2.7].

THEOREM 2.5. (Binding and Li) *Let $A_1,A_2,A_3 \in H_n$ with $n \geqslant 3$ and $\xi \in W(A_1,A_2,A_3)$. Then ξ is a sharp point if and only if there is a unitary matrix $U \in \mathbb{C}_{n\times n}$ such that*

$$U^*A_iU = \xi_i I_m \oplus B_i, \quad i = 1,2,3 \tag{2.1}$$

with $\xi = (\xi_1,\xi_2,\xi_3) \notin \text{conv}\, W(B_1,B_2,B_3)$.

Similar to the $W(A)$ case, for each $\xi \in W(A_1,A_2,A_3)$ we define

$$W_\xi = W_\xi(A_1,A_2,A_3) := \{x \in \mathbb{C}^n : (x^*A_1x, x^*A_2x, x^*A_3x) = \xi x^*x\}.$$

Notice that W_ξ is homogenous so that $\langle W_\xi \rangle = W_\xi + W_\xi$. It is natural to ask whether Theorem 1.2 (1) can be extended to $W(A_1,A_2,A_3)$. Unfortunately, for Example 2.3 with $n = 3$, W_0 is a 1-dimensional subspace, but 0 is clearly not an extreme point of the unit ball $W(A_1,A_2,A_3)$. But the problem can be resolved in the following theorem. It generalizes the first two parts of Theorem 1.2.

THEOREM 2.6. *Let $A_1,A_2,A_3 \in H_n$ with $n \geqslant 3$ and $\xi \in W(A_1,A_2,A_3)$. Then*

1. *ξ is an extreme point if and only if ξ is a boundary point and W_ξ is a subspace of $\mathbb{C}^n$. In particular, if $W(A_1,A_2,A_3)$ is degenerate, then ξ is an extreme point if and only if W_ξ is a subspace of $\mathbb{C}^n$.*
2. *if ξ is a non-extreme boundary point and if P is a supporting plane of $W(A_1,A_2,A_3)$ at ξ, then*

$$\langle W_\xi \rangle \subseteq \cup_{z \in P\cap W(A_1,A_2,A_3)} W_z. \tag{2.2}$$

In addition, if $P \cap W(A_1,A_2,A_3)$ is
(i) a flat convex set containing ξ as a relative interior point, or
(ii) a line segment,
then

$$\langle W_\xi \rangle = \cup_{z \in P\cap W(A_1,A_2,A_3)} W_z; \tag{2.3}$$

(iii) a flat convex set S in which ξ is not a relative interior point of S, then

$$\langle W_\xi \rangle = \cup_{z\in L} W_z,$$

where L is the longest line segment in $W(A_1,A_2,A_3)$ that contains ξ.
In case of (i) or (ii), $\langle W_\xi \rangle = \mathbb{C}^n$ if and only if $W(A_1,A_2,A_3) \subseteq P$.

Proof. (1) Evidently all extreme points of $W(A_1,A_2,A_3)$ are boundary points. Suppose that ξ is a boundary point of $W(A_1,A_2,A_3)$. Since $W(A_1,A_2,A_3)$ is convex [2], there is a supporting plane P of $W(A_1,A_2,A_3)$ at ξ. Let $p=(p_1,p_2,p_3)$ be a unit vector perpendicular to P. Project $W(A_1,A_2,A_3)$ onto $\langle p\rangle$. So $\eta := p^T\xi = p_1\xi_1+p_2\xi_2+p_3\xi_3$ is an extreme point of the classical numerical range $W(\hat{A})$ (a line segment), where $\hat{A} := p_1A_1+p_2A_2+p_3A_3$. So η must be a maximal or minimal eigenvalue of the Hermitian matrix $\hat{A}$. Thus $M_\eta(\hat{A})$ is an eigenspace of $\hat{A}$. In addition if ξ is an extreme point of $W(A_1,A_2,A_3)$, we have $W_\xi(A_1,A_2,A_3) = M_\eta(\hat{A})$.

Suppose that ξ is a boundary point and W_ξ is a subspace of $\mathbb{C}^n$. If ξ were not an extreme point, there would exist distinct $\alpha,\beta\in W(A_1,A_2,A_3)$ such that $\xi\in(\alpha,\beta)$. Let $u,v\in\mathbb{S}^{n-1}$ such that $\alpha=f(u)$, $\beta=f(v)$ and let $\hat{A}_i$ $(i=1,2,3)$ denote the compression of A_i onto the 2-dimensional subspace $\langle u,v\rangle$. So $f(u),f(v)\in W(\hat{A}_1,\hat{A}_2,\hat{A}_3)\subseteq W(A_1,A_2,A_3)$ and ξ would be contained in the convex hull of the ellipsoid $W(\hat{A}_1,\hat{A}_2,\hat{A}_3)$. Since ξ is a boundary point of $W(A_1,A_2,A_3)$, this forces $W(\hat{A}_1,\hat{A}_2,\hat{A}_3)$ to be an elliptical disk (possibly degenerate but not a point since $\alpha\neq\beta$). Hence $\xi\in W(\hat{A}_1,\hat{A}_2,\hat{A}_3)$ and by Theorem 2.2(1), there would exist two linearly independent unit vectors $x,y\in\langle u,v\rangle$ such that $f(x)=f(y)=\xi$. But W_ξ is a subspace and clearly $x,y\in W_\xi$. So $u\in\langle u,v\rangle=\langle x,y\rangle\subseteq W_\xi$ and we would have $\alpha=f(u)=\xi$ which is absurd.

(2) Suppose that ξ is a non-extreme boundary point. Let P be a supporting plane of $W(A_1,A_2,A_3)$ at ξ. Referring to the first paragraph of the proof of (1), $M_\eta(\hat{A})$ is an eigenspace of $\hat{A}$ and

$$M_\eta(\hat{A}) = \cup_{z\in P\cap W(A_1,A_2,A_3)} W_z.$$

Since $\xi\in P$, $W_\xi\subseteq\cup_{z\in P\cap W(A_1,A_2,A_3)}W_z$ we have (2.2)

$$\langle W_\xi\rangle\subseteq\cup_{z\in P\cap W(A_1,A_2,A_3)}W_z.$$

On the other hand, pick arbitrary $z\in P\cap W(A_1,A_2,A_3)$ with $z\neq\xi$.
(i) If $P\cap W(A_1,A_2,A_3)$ is a flat convex set in which ξ is an interior point, then there exists $z'\neq z$ and $\xi\in(z,z')\subseteq P\cap W(A_1,A_2,A_3)$. Pick $x\in W_z$ and $x'\in W_{z'}$. Clearly x,x' are linearly independent since $z\neq z'$. Let $\hat{A}_1,\hat{A}_2,\hat{A}_3$ be the compressions of A_1,A_2,A_3 onto $\langle x,x'\rangle$ respectively. As an ellipsoid containing z,z', $W(\hat{A}_1,\hat{A}_2,\hat{A}_3)\subseteq W(A_1,A_2,A_3)$ must be degenerate. Thus by Theorem 2.2(1b), there are two linearly independently vectors $u,v\in\langle x,x'\rangle\cap\mathbb{S}^{n-1}$ such that $f(u)=f(v)=\xi$. But $x\in\langle x,x'\rangle=\langle u,v\rangle\subseteq W_\xi+W_\xi=\langle W_\xi\rangle$. So $W_z\subseteq\langle W_\xi\rangle$ for all $z\in P\cap W(A_1,A_2,A_3)$. So we have the other inclusion

$$\langle W_\xi\rangle\supseteq\cup_{z\in P\cap W(A_1,A_2,A_3)}W_z.$$

(ii) If $P\cap W(A_1,A_2,A_3)$ is a line segment, then there exists $z'\neq z$ and ξ is in the open segment (z,z'). Similar to (i), $W(\hat{A}_1,\hat{A}_2,\hat{A}_3)\subseteq W(A_1,A_2,A_3)$ is a line segment and apply Theorem 2.2(1b).
(iii) Project $W(A_1,A_2,A_3)$ onto the hyperplane H (spanned by $p=(p_1,p_2,p_3)$ and (q_1,q_2,q_3)) that is orthogonal to the line L. So L is projected into a point $\eta\in H$. Hence η is an extreme point of $W(\hat{A}_1+i\hat{A}_2)$ where $\hat{A}_1=p_1A_1+p_2A_2+p_3A_3$ and $\hat{A}_2=q_1A_1+q_2A_2+q_3A_3$. So M_η is a subspace by Theorem 1.2(1). Moreover $W_\xi\subseteq$

$\cup_{z\in L} W_z = M_\eta$ so that $\langle W_\xi \rangle \subseteq \cup_{z\in L} W_z$. Then use the argument in (ii) to have $\langle W_\xi \rangle \supseteq \cup_{z\in L} W_z$ since $\xi \in L$. $\square$

COROLLARY 2.7. *Let $A_1, A_2, A_3 \in H_n$ with $n \geqslant 4$ and $\xi \in W(A_1, A_2, A_3)$. If $w_{A_1,A_2,A_3}(\xi) = 1$, then ξ is an extreme point.*

EXAMPLE 2.8. With respect to Theorem 2.6(2)(iii) it is possible that there is only one supporting plane P at ξ such that $S := P \cap W(A_1, A_2, A_3)$ is a flat convex set and ξ is not in the relative interior of $S \subseteq \mathbb{R}^3$. For example, if

$$\begin{aligned} A_1 &= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus 0 \oplus \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus 0 = B \oplus 0 \oplus B \oplus 0, \\ A_2 &= \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \oplus 0 \oplus \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \oplus 0 = C \oplus 0 \oplus C \oplus 0, \\ A_3 &= I_3 \oplus (-I_3), \end{aligned}$$

then $W(A_1, A_2, A_3) = \operatorname{conv}\{W(B, C, I_2), W(B, C, -I_2), L\}$, where L is the line segment joining $(0,0,1)$ and $(0,0,-1)$. Notice that $W(B, C, \pm I_2)$ is the circular disk

$$\{(x, y, \pm 1) : (x-1)^2 + (y-1)^2 = 1\}$$

so that $W(A_1, A_2, A_3)$ is the convex hull of the cylinder

$$K := \{(x, y, z) : -1 \leqslant z \leqslant 1, (x-1)^2 + (y-1)^2 = 1\}$$

and L. The point $\xi := (1,0,0) \in K$ lies on the edge of the flat portion

$$S := \{(x, 0, z) : 0 \leqslant x \leqslant 1, -1 \leqslant z \leqslant 1\} \subseteq W(A_1, A_2, A_3)$$

but is not an extreme point. The vector $x \in \mathbb{S}^5$ such that $\xi = (x^*A_1x, x^*A_2x, x^*A_3x)$ must be of the form $x = \frac{1}{2}(u, v)$, $u, v \in \mathbb{S}^2$ in view of the zero third coordinate. So $u = \rho_1(1, i, 0)$ and $v = \rho_2(1, i, 0)$, $|\rho_1| = |\rho_2| = 1$. Thus $x = \frac{1}{2}(\rho_1, i\rho_2, 0, \rho_2, i\rho_2, 0)$. So

$$W_\xi = \langle W_\xi \rangle = \operatorname{span}\{(1, i, 0, 0, 0, 0), (0, 0, 0, 1, i, 0)\}.$$

For any point $\eta := (t, 0, 0)$ for $0 < t < 1$, W_η is not contained in W_ξ. So (2.3) does not hold though (2.2) is true.

3. Joint numerical range of two real symmetric matrices

Brickman [5] (also see [12]) studied the real analog of the numerical range of $A \in \mathbb{C}_{n\times n}$:

$$V(A) = \{x^T A x : x \in \mathbb{R}^n, x^T x = 1\}$$

and proved that $V(A)$ is convex when $n \geqslant 3$. In addition $V(A)$ is an ellipse (possibly degenerate) when $n = 2$. Indeed [12]

$$W(A) = V(\hat{A})$$

where

$$\hat{A} = \begin{pmatrix} A & iA \\ -iA & A \end{pmatrix}.$$

So convexity of $W(A)$ follows from the convexity of $V(\hat{A})$ when $n \geqslant 2$. Clearly

$$V(A) = V(\frac{A+A^T}{2})$$

so that we can restrict our study to symmetric A. Moreover if $A = A^T$, then $W(A) = \operatorname{conv} V(A)$ [12] and in particular, if $n \geqslant 3$, then $W(A) = V(A)$.

See [13] for an interesting unified treatment for $W(A)$, $W(A_1, A_2, A_3)$ and $V(A)$; [11] for more general notions in the context of semisimple Lie algebras; and [10] for related results.

The following is a list of some basic properties of $V(A)$, similar to those of $W(A)$.

LEMMA 3.1. *Let $A \in \mathbb{C}_{n\times n}$ be symmetric.*

1. $V(\alpha A + \beta I_n) = \alpha V(A) + \beta$, $\alpha, \beta \in \mathbb{C}$.
2. $V(O^T A O) = V(A)$ *for any orthogonal matrix* $O \in \mathrm{O}(n)$.
3. $V(A) = \{\lambda\}$ *if and only if* $A = \lambda I_n$.
4. *If* $B \in \mathbb{C}_{m\times m}$ *is a principal submatrix of* A, *then* $V(B) \subseteq V(A)$.
5. *(McIntosh* [12]*)* $\operatorname{conv} V(A) = W(A)$.

DEFINITION 3.2. Let $A \in \mathbb{C}_{n\times n}$ be symmetric and $\xi \in V(A)$. The multiplicity of ξ, denoted by $\nu_A(\xi)$, is the maximal number of linearly independent vectors $x \in \mathbb{S}^{n-1}_{\mathbb{R}} := \mathbb{S}^{n-1} \cap \mathbb{R}^n$ such that $x^T A x = \xi$.

The following result and its proof are similar to Theorem 2.2.

THEOREM 3.3. *Let $A \in \mathbb{C}_{n\times n}$ be symmetric.*

1. *Suppose* $n = 2$ *and write* $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ *with* $a_{12} = a_{21}$.
   - (a) $V(A) = \{\alpha\}$ *if and only if* $A = \alpha I_2$ *for some* $\alpha \in \mathbb{C}$. *In this case,* $\nu_A(\alpha) = 2$.
   - (b) $V(A)$ *is a nondegenerate line segment* $[\alpha, \beta]$ *if and only if either* $a_{11} = a_{22}$ *or* $a_{12} = 0$, *but not both. Moreover* $\nu_A(\alpha) = \nu_A(\beta) = 1$ *and* $\nu_A(\xi) = 2$ *if* $\xi \in (\alpha, \beta)$.
   - (c) $V(A)$ *is a nondegenerate ellipse if and only if both* $a_{11} \neq a_{22}$ *and* $a_{12} \neq 0$. *In this case,* $\nu_A(\xi) = 1$ *for all* $\xi \in V(A)$.
2. *When* $n \geqslant 3$, *if* $\xi \in \operatorname{Int}_R V(A)$, *then* $\nu_A(\xi) \geqslant n-2$ *and may not be a constant. The lower bound* $n-2$ *is best possible.*

Proof. (1) When $n=2$,

$$\begin{aligned} V(A) &= \{a_{11}\cos^2\theta + (a_{12}+a_{21})\cos\theta\sin\theta + a_{22}\sin^2\theta : 0 \leqslant \theta < 2\pi\} \\ &= \{\frac{(a_{11}-a_{22})}{2}\cos 2\theta + \frac{(a_{12}+a_{21})}{2}\sin 2\theta + \frac{(a_{11}+a_{22})}{2} : 0 \leqslant \theta < 2\pi\} \end{aligned}$$

which is an ellipse with center $(\operatorname{tr} A)/2$. We only need to consider the multiplicities for the following cases since the rest is straightforward computation.

(b) $V(A)$ is a nondegenerate line segment $[\alpha,\beta]$. If ξ is one of the endpoints, it corresponds to two values of $\theta \in [0,2\pi)$ of difference π. So the corresponding x are negative to each other. Hence $v_A(\xi)=1$. If $\xi \in (\alpha,\beta)$, there are four desired values of θ which yield two linearly independent vectors, and hence $v_A(\xi)=2$.

(c) $V(A)$ is a nondegenerate ellipse. Each $\xi \in V(A)$ corresponds to two values of $\theta \in [0,2\pi)$ of difference π. Hence $v_A(\xi)=1$.

(2) The statement is trivial for $n=3$. Suppose $n \geqslant 4$ and $\xi \in \operatorname{Int}_R V(A)$. Suppose on the contrary that $v_A(\xi)=k<n-2$. Let $g : \mathbb{S}_{\mathbb{R}}^{n-1} \to \mathbb{C}$ be the map defined by

$$g(x) := x^T A x, \quad x \in \mathbb{S}_{\mathbb{R}}^{n-1}.$$

Let $\{x_1,\ldots,x_k\} \subseteq \mathbb{S}_{\mathbb{R}}^{n-1}$ be a (maximal) linearly independent set such that $\xi = g(x_i)$, $i=1,\ldots,k$. Choose $u \in \mathbb{S}_{\mathbb{R}}^{n-1}$ such that $x_1,\ldots,x_k,u$ are linearly independent and clearly $\xi \neq g(u)$. Because $\xi \in \operatorname{Int}_R V(A)$ and $V(A)$ is convex [12], there is $v \in \mathbb{S}_{\mathbb{R}}^{n-1}$ such that

(a) $g(v) \neq \xi$ and $g(v) \neq g(u)$,

(b) the line segment $L = [g(u),g(v)] \subseteq V(A)$, and

(c) $\xi \in L$.

Since $g(u) \neq g(v)$, u,v are linearly independent. Since $k<n-2$, there is $w \in \mathbb{S}_{\mathbb{R}}^{n-1}$ and $w \notin \langle u,v,x_1,\ldots,x_k\rangle$. Let $\hat{A}$ denote the compression of A onto the three dimensional subspace $\langle u,v,w\rangle$. Since $V(\hat{A})$ is convex, there would exist a unit vector $y \in \langle u,v,w\rangle$ and $g(y)=\xi$. Write $y=\alpha u+\beta v+\gamma w$ for $\alpha,\beta,\gamma \in \mathbb{R}$. Notice that α and γ cannot be both zero, otherwise $g(y)=g(v)\neq\xi$. But then $y,x_1,\ldots,x_k$ would be linearly independent, a contradiction.

The following example shows that the lower bound $n-2$ is best possible and $v_A(\xi)$ may not be a constant. $\square$

EXAMPLE 3.4. Let $A = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix} \oplus 0 \in \mathbb{C}_{n\times n}$, where $n \geqslant 3$. Then $V(A)$ is the unit disk since it is the direct sum of $V(B)$ and the origin where $B = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$. Then (a) and (b) of the following can be deduced immediately and (c) can be computed directly:

(a) $v_A(0)=n-2$,

(b) $v_A(\xi)=n-1$ if $0<|\xi|<1$, and

(c) $v_A(\xi)=1$ if $|\xi|=1$.

DEFINITION 3.5. Let $A \in \mathbb{C}_{n\times n}$ be symmetric and $\xi \in V(A)$. We define

$$V_\xi = V_\xi(A) := \{x \in \mathbb{R}^n : x^T Ax = \xi x^T x\}.$$

The ideas of conical points in [4] can be applied to $V(A)$ since $V(A)$ can be identified as the joint numerical range of A_1 and A_2:

$$\{(x^T A_1 x, x^T A_2 x) : x^T x = 1, x \in \mathbb{R}^n\} \subseteq \mathbb{R}^2,$$

where $A = A_1 + iA_2$ and A_1 and A_2 are real symmetric matrices. Adapting the approach in [4] yields the following result. We now provide a different proof and remark that the approach applies to Theorem 2.5.

THEOREM 3.6. (Binding and Li) *Let $A \in \mathbb{C}_{n\times n}$ be symmetric and $\xi \in V(A)$. Then ξ is a sharp point if and only if there is an orthogonal matrix $O \in \mathrm{O}(n)$ such that $O^T AO = \xi I_m \oplus B$, with $\xi \notin \operatorname{conv} V(B)$.*

Proof. Suppose that there is an orthogonal matrix $O \in \mathrm{O}(n)$ such that $O^T AO = \xi I_m \oplus B$, with $\xi \notin \operatorname{conv} V(B)$ and $B \in \mathbb{C}_{(n-m)\times(n-m)}$ is symmetric. By Lemma 3.1

$$V(A) = V(O^T AO) = \operatorname{conv}\{\xi, V(B)\}.$$

Since ξ is not contained in the compact convex set $\operatorname{conv} V(B)$, ξ is a sharp point of $V(A)$.

Conversely suppose that ξ is a sharp point of $V(A)$. Without loss of generality, we may assume that $\xi = a_{11}$ otherwise we perform an orthogonal similarity on A. For each $i = 2,\ldots,n$, the 2×2 principal submatrix $A_i := \begin{pmatrix} a_{11} & a_{1i} \\ a_{i1} & a_{ii} \end{pmatrix}$ of A satisfies $V(A_i) \subseteq V(A)$. Since $\xi = a_{11}$ is a sharp point, $V(A_i)$ must be a line segment (possibly degenerate). By Lemma 3.3, $a_{1i} = a_{i1}$ must be zero. Thus $A = \xi \oplus \hat{A}$, where $\hat{A} \in \mathbb{C}_{(n-1)\times(n-1)}$ is symmetric, and in particular ξ is an eigenvalue of A. So $V(A) = \operatorname{conv}\{\xi, V(\hat{A})\}$. Since ξ is a sharp point, ξ is not contained in the relative interior of $\operatorname{conv} V(\hat{A})$. So ξ is either in $V(\hat{A})$ or not (even when $\hat{A} \in \mathbb{C}_{2\times 2}$ with $V(\hat{A})$ an ellipse). If $\xi \notin V(\hat{A})$, it forces that $\xi \notin \operatorname{conv} V(\hat{A})$ since ξ is a sharp point and we are done. Otherwise, repeat the argument on $\hat{A}$ to arrive at the desired conclusion. □

Similar to $W(A_1,A_2,A_3)$, Theorem 1.2(1) cannot be extended to $V(A)$ by observing Example 3.4 with $n = 3$: V_0 is a 1-dimensional subspace, but 0 is not an extreme point of the unit disk $V(A)$.

THEOREM 3.7. *Let $A \in \mathbb{C}_{n\times n}$ be symmetric with $n \geqslant 3$ and $\xi \in V(A)$. Then*

1. *ξ is an extreme point if and only if ξ is a boundary point and V_ξ is a subspace of $\mathbb{R}^n$.*

2. *if ξ is a non-extreme boundary point, then*

$$\langle V_\xi \rangle = \cup_{z\in L} V_z,$$

where L is the supporting plane of $V(A)$, passing through ξ. In this case $\langle V_\xi \rangle = \mathbb{R}^n$ if and only if $V(A) \subseteq L$.

Proof. (1) All extreme points of $V(A)$ are boundary points. Suppose that $\xi = \xi_1 + i\xi_2$ ($\xi_1, \xi_2 \in \mathbb{R}$) is an extreme point of $V(A)$. Since $V(A)$ is convex [5, 12], there is a supporting line L of $V(A)$ at ξ. Let $p := p_1 + ip_2 \in \mathbb{C}$ ($p_1, p_2 \in \mathbb{R}$) be a unit vector perpendicular to L. Project $V(A)$ onto $\langle p \rangle$. If $A = A_1 + iA_2$ is the Hermitian decomposition, then $\eta := p_1\xi_1 + p_2\xi_2$ is an extreme point of $V(p_1A_1 + p_2A_2)$ (a line segment), where $p_1A_1 + p_2A_2$ is clearly real symmetric. By the spectral theorem for real symmetric matrices, $V_\eta(p_1A_1 + p_2A_2)$ is the eigenspace of $p_1A_1 + p_2A_2$. Since ξ is an extreme point of $V(A)$, $V_\xi(A) = V_\eta(p_1A_1 + p_2A_2)$ and is a subspace of $\mathbb{R}^n$.

Suppose that $\xi \in V(A)$ is a boundary point and V_ξ is a subspace of $\mathbb{R}^n$. If ξ were not an extreme point, there would exist distinct $\alpha, \beta \in V(A)$ such that $\xi \in (\alpha, \beta)$. Let $u, v \in \mathbb{S}_{\mathbb{R}}^{n-1}$ such that $\alpha = g(u)$, $\beta = g(v)$ and let $\hat{A}$ denote the compression of A onto the 2-dimensional subspace $\langle u, v \rangle$. So $g(u), g(v) \in V(\hat{A}) \subseteq V(A)$ and ξ would be contained in the convex hull of the ellipse $V(\hat{A})$. Since ξ is a boundary point of $V(A)$, this forces $V(\hat{A})$ to be a line segment (but not a point since $\alpha \neq \beta$). Hence $\xi \in V(\hat{A})$ and thus there would exist two linearly independent unit vectors $x, y \in \langle u, v \rangle$ such that $g(x) = g(y) = \xi$. Thus $\langle u, v \rangle = \langle x, y \rangle$. But V_ξ is a subspace and clearly $x, y \in V_\xi$. So $u \in \langle x, y \rangle \subseteq V_\xi$ and we would have $\alpha = g(u) = \xi$ which is absurd.

(2) Suppose that ξ is a non-extreme boundary point of $V(A)$. Let L be the supporting line of $V(A)$ at ξ. Referring to the first paragraph of the proof of (1), $V_\eta(p_1A_1 + p_2A_2)$ is the eigenspace of $p_1A_1 + p_2A_2$ and $V_\eta(p_1A_1 + p_2A_2) = \cup_{z \in L \cap V(A)} V_z$. Since $\xi \in L$, clearly $V_\xi \subseteq \cup_{z \in P \cap V(A)} V_z$ and we have

$$\langle V_\xi \rangle \subseteq \cup_{z \in P \cap V(A)} V_z.$$

On the other hand, suppose $z \in L \cap V(A)$ and $z \neq \xi$. Then there exists $z' \neq z$ and ξ is in the open segment (z, z'). Pick $x \in V_z$ and $x' \in V_{z'}$. Clearly x, x' are linearly independent since $z \neq z'$. Let $\hat{A}$ be the compressions of A onto $\langle x, x' \rangle$ respectively. Then the ellipse $V(\hat{A}) \subseteq V(A)$ must degenerate since $\xi \in [z, z'] \subseteq W(\hat{A})$ and is a boundary point of $V(A)$. Thus by Theorem 3.3(1b), there are two linearly independently vectors $u, v \in \langle x, x' \rangle \cap \mathbb{S}_{\mathbb{R}}^{n-1}$ such that $f(u) = f(v) = \xi$. But $x \in \langle x, x' \rangle = \langle u, v \rangle = V_\xi + V_\xi = \langle V_\xi \rangle$. So $V_z \subseteq \langle V_\xi \rangle$ for all $z \in L \cap V(A)$. So we have the other inclusion $\langle W_\xi \rangle \supseteq \cup_{z \in L \cap V(A)} V_z$. $\square$

We remark that the above technique can be used to prove Theorem 1.2 and is different from the approach of Embry in [8].

COROLLARY 3.8. *Let $V \in \mathbb{C}_{n \times n}$ with $n \geqslant 4$ and $\xi \in V(A)$. If $v_A(\xi) = 1$, then ξ is an extreme point.*

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In section 3, $W(A) = W(\frac{1}{2}(\hat{A} + \hat{A}^T))$ for an any $A \in \mathbb{C}_{n \times n}$. The referee asked the following question: Let A be an $n \geqslant 3$. Does there exist an symmetric complex matrix $S \in \mathbb{C}_{n \times n}$ satisfying $W(A) = W(S)$? Let $A =$.

REFERENCES

[1] Y. H. AU-YEUNG AND Y. T. POON, *A remark on the convexity and positive definiteness concerning Hermitian matrices*, Southeast Asian Bull. Math., **3** (1979), 85–92.

[2] Y. H. AU-YEUNG AND N. K. TSING, *An extension of the Hausdorff-Toeplitz theorem*, Proc. Amer. Math. Soc., **89** (1983), 215–218.

[3] Y. H. AU-YEUNG AND N. K. TSING, *Some theorems on the numerical range*, Linear and Multilinear Algebra, **15** (1984), 215–218.

[4] P. L. BINDING AND C. K. LI, *Joint ranges of Hermitian matrices and simultaneous diagonalization*, Linear Algebra Appl., **151** (1991), 157–168.

[5] L. BRICKMAN, *On the field of values of a matrix*, Proc. Amer. Math. Soc., **12** (1961), 61–66.

[6] C. DAVIS, *The Toeplitz-Hausdorff theorem explained*, Canad. Math. Bull., **14** (1971), 245–246.

[7] W. F. DONOGHUE, JR., *On the numerical range of a bounded operator*, Michigan Math. J., **4** (1957), 261–263.

[8] M. R. EMBRY, *The numerical range of an operator*, Pacific J. Math., **32** (1970), 647–650.

[9] R. A. HORN AND C. R. JOHNSON, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[10] C. K. LI AND Y. T. POON, *Convexity of the joint numerical range*, SIAM J. Matrix Anal. Appl., **21** (1999), 668–678.

[11] C. K. LI AND T. Y. TAM, *Numerical ranges arising from simple Lie algebras*, J. Canad. Math. Soc., **52** (2000), 141–171.

[12] A. MCINTOSH, *The Toeplitz-Hausdorff theorem and ellipticity conditions*, Amer. Math. Monthly, **85** (1978), 475–477.

[13] Y. T. POON, *Generalized numerical ranges, joint positive definiteness and multiple eigenvalues*, Proc. Amer. Math. Soc., **125** (1997), 1625–1634.

[14] J. G. STAMPFLI, *Extreme points of the numerical range of hyponormal operators*, Michigan Math. J., **13** (1966), 87–89.

[15] F. UHLIG, *An inverse field of values problem*, Inverse Problems, **24** (2008), 055019, 19 pages.

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