Math 5050/6050 Key to Chapter 13

Exercises

1. Follows from the definition of $\text{vec}\,(C)$ in Definition 13.17.

2. (P_1)

$$\begin{aligned}(A\otimes B)(A^+\otimes B^+)(A\otimes B) &= AA^+A\otimes BB^+B \qquad \text{by Theorem 13.3}\\ &= A\otimes B\end{aligned}$$

   (P_2) Similar.

   (P_3)

$$\begin{aligned}[(A\otimes B)(A^+\otimes B^)]^T &= (AA^T\otimes BB^+)^T\\ &= (AA^+)^T\otimes(BB^+)^T \qquad \text{by Theorem 13.4}\\ &= AA^+\otimes BB^+\end{aligned}$$

   (P_4) Similar.

3. Let $A\in\mathbb{R}^{m\times n}$, $B\in\mathbb{R}^{p\times q}$ and $C\in\mathbb{R}^{m\times q}$. $AXB=C$ means $\text{vec}\,(AXB)=\text{vec}\,C$, i.e., by Theorem 13.26, $(B^T\otimes A)\text{vec}\,X=\text{vec}\,C$. So $AXB=C$ has solution for all C if and only if $(B^T\otimes A)x=c$ has a solution for all $c:=\text{vec}\,C$, where $x:=\text{vec}\,X$. In other words, $R(B^T\otimes A)=\mathbb{R}^{mq}$, or equivalently, $\text{rank}\,(B^T\otimes A)=mq$. However, by Corollary 13.11, $mq=\text{rank}\,(B^T\otimes A)=\text{rank}\,B^T\,\text{rank}\,A=\text{rank}\,A\,\text{rank}\,B\le mq$ since $\text{rank}\,A\le m$ and $\text{rank}\,B\le q$. So the necessary and sufficient condition for the existence of solution is that A has full row rank and B has full column rank. In addition, if A has full column rank and B has full row rank, i.e., $m=n=\text{rank}\,A$ and $p=q=\text{rank}\,B$, then A and B are nonsingular. By (13.15) the solution is unique, i.e., $A=A^{-1}CB^{-1}$.

4. Follows directly from Theorem 13.26 with the observation $\text{vec}\,(\alpha A+\beta B)=\alpha\text{vec}\,A+\beta\text{vec}\,B$, i.e., $\text{vec}\,:\mathbb{R}^{m\times n}\to\mathbb{R}^{mn}$ is a linear map. Indeed it is an isomorphism.

5.

$$\begin{aligned}(x^T\otimes y)^T &= x\otimes y^T \qquad \text{by Theorem 13.4}\\ &= xy^T \qquad \text{Example 13.2(5) on p.140}\end{aligned}$$

   So $x^T\otimes y=(xy^T)^T=yx^T$.

6. (a) $||A\otimes B||_2=\sigma_1(A\otimes B)$. By theorem 13.10, $\sigma_1(A\otimes B)$ is $\sigma_1(A)\sigma_1(B)$. So $||A\otimes B||_2=||A||_2||B||_2$.

   (b)

$$\begin{aligned}||A\otimes B||_F^2 &= \text{tr}\,(A\otimes B)^T(A\otimes B) \qquad \text{definition}\\ &= \text{tr}\,(A^T\otimes B^T)(A\otimes B) \qquad \text{Theorem 13.4}\\ &= \text{tr}\,(A^TA\otimes B^TB) \qquad \text{Theorem 13.3}\\ &= (\text{tr}\,A^TA)(\text{tr}\,B^TB) \qquad \text{Corollary 13.13(1)}\\ &= ||A||_F^2||B||_F^2\end{aligned}$$

   So, $||A\otimes B||_F=||A||_F||B||_F$.

7. (a) By induction with Theorem 13.3.

(b)

$$\begin{aligned} e^{I\otimes A} &= \sum_{k=0}^{\infty} \frac{(I\otimes A)^k}{k!} = \sum_{k=0}^{\infty} \frac{I\otimes A^k}{k!} \qquad \text{by (a)} \\ &= I\otimes \sum_{k=0}^{\infty} \frac{A^k}{k!} = I\otimes e^A \end{aligned}$$

Similar $e^{B\otimes I} = e^B \otimes I$.

(c) $(I\otimes A)(B\otimes I) = B\otimes A$ by Theorem 13.3 and $(B\otimes I)(I\otimes A) = B\otimes A$ by Theorem 13.3. So $I\otimes A$ and $B\otimes I$ commute.

(d) $e^{A\oplus B} = e^{(I\otimes A + B\otimes I)} = e^{I\otimes A}e^{B\otimes I} = e^A\otimes e^B$ by (c) and p.110 #4

or $e^{A\oplus B} = e^{(I\otimes A + B\otimes I)} = e^{B\otimes I}e^{I\otimes A} = (e^B\otimes I)(I\otimes e^A) = e^B\otimes e^A$.

Remark: $A\oplus B$ is not a good notation (p.142).

8. Direct substitution verifies that X_s and X_{ns} are solutions. The eigenvalues of A are ± 1 and the eigenvalues of $-A^T$ are ± 1 as well. So the condition in Theorem 13.21 that A and $-A^T$ do not have the same eigenvalues is not met.

9. (a) Suppose $C - XA + DX - XBX = 0$ (Riccati equation). Let $T = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}$. Then $T^{-1} = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}$ (p.48).

$$\begin{aligned} T^{-1}ST &= \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} I & 0 \\ X & I \end{bmatrix} \\ &= \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}\begin{bmatrix} A+BX & B \\ C+DX & D \end{bmatrix} \\ &= \begin{bmatrix} A+BX & B \\ -XA-XBX+C+DX & -XB+D \end{bmatrix} \\ &= \begin{bmatrix} A+BX & B \\ 0 & D-XB \end{bmatrix} \end{aligned}$$

(b) T would then be upper triangular and similar result holds.

10. (a)

$$\begin{aligned} T^{-1}ST = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} &\Leftrightarrow ST = T\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \\ &\Leftrightarrow \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}\begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \\ &\Leftrightarrow AY - YD = -B \qquad \text{by comparing the (1,2) block} \end{aligned}$$

(b) Find $R = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix}$ so that $R^{-1}SR = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$. Then S is similar to $\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}$ via R if and only if S^T is similar to $\begin{bmatrix} A^T & 0 \\ 0 & D^T \end{bmatrix}$ via $T = R^T$. If $A^TY^T - Y^TD^T = -C^T$, i.e., Y satisfies the Sylvester equation $YA - DY = -C$

Math 5050/6050 Supplementary notes to Chapter 13

See Some Theorems on Matrix Differentiation with Special Reference to Kronecker Matrix Products by H. Neudecker Journal of the American Statistical Association, Vol. 64, No. 327 (Sep., 1969), pp. 953-963.

Section 13.2 The proof of Theorem 13.12 is just a basic idea and does not take care of multiplicities of eigenvalues. Better use Jordan canonical form of Schur triangularization form in Chapter 9, alike the remark on Theorem 13.16.

Section 13.3 The first part of Theorem 13.21 is a Corollary of Theorem 13.18 in which $B = A^T$. Suppose in addition that C is symmetric. Let X be the unique solution to $AX + XA^T = C$. Then $X^TA^T + AX = C^T = C$, i.e., X^T is also a solution to the equation. But the solution is unique so that $X = X^T$, i.e., X is symmetric.

Proof of Theorem 13.26: Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$ and $C \in \mathbb{R}^{p\times q}$. The ith column of AB is Ab_i and the jth column of ABC is

$$ABc_j = \sum_{i=1}^{p} c_{ij}Ab_i = (c_j^T \otimes A)\text{vec}\, B$$

so that $\text{vec}\,(ABC) = (C^T \otimes A)\text{vec}\, B$.

On p.148, Use (P1)-(P4) to check $(M \otimes N)^+ = M^+ \otimes N^+$.