Math 5050/6050 Key to Chapter 13

Exercises

1. Follows from the definition of $\text{vec } (C)$ in Definition 13.17.

2. $(P_1)$

\[(A \otimes B)(A^+ \otimes B^+)(A \otimes B) = AA^+ A \otimes BB^+ B \quad \text{by Theorem 13.3}
\]

\[= A \otimes B
\]

$(P_2)$ Similar.

$(P_3)$

\[[(A \otimes B)(A^+ \otimes B^+)]^T = (AA^T \otimes BB^+)^T
\]

\[= (AA^+)^T \otimes (BB^+)^T \quad \text{by Theorem 13.4}
\]

\[= AA^+ \otimes BB^+
\]

$(P_4)$ Similar.

3. Let $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$ and $C \in \mathbb{R}^{m \times q}$. $AXB = C$ means $\text{vec } (AXB) = \text{vec } C$, i.e., by Theorem 13.26, $(B^T \otimes A)\text{vec } X = \text{vec } C$. So $AXB = C$ has solution for all $C$ if and only if $(B^T \otimes A)x = c$ has a solution for all $c := \text{vec } C$, where $x := \text{vec } X$. In other words, $R(B^T \otimes A) = \mathbb{R}^{mq}$, or equivalently, $\text{rank } (B^T \otimes A) = mq$. However, by Corollary 13.11, $mq = \text{rank } (B^T \otimes A) = \text{rank } B^T \text{rank } A = \text{rank } A \text{rank } B \leq mq$ since $\text{rank } A \leq m$ and $\text{rank } B \leq q$. So the necessary and sufficient condition for the existence of solution is that $A$ has full row rank and $B$ has full column rank. In addition, if $A$ has full column rank and $B$ has full row rank, i.e., $m = n = \text{rank } A$ and $p = q = \text{rank } B$, then $A$ and $B$ are nonsingular. By (13.15) the solution is unique, i.e., $A = A^{-1}CB^{-1}$.

4. Follows directly from Theorem 13.26 with the observation $\text{vec } (\alpha A + \beta B) = \alpha \text{vec } A + \beta \text{vec } B$, i.e., $\text{vec } : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{mn}$ is a linear map. Indeed it is an isomorphism.

5. \[ (x^T \otimes y)^T = x \otimes y^T \quad \text{by Theorem 13.4}
\]

\[= xy^T \quad \text{Example 13.2(5) on p.140}
\]

So $x^T \otimes y = (xy^T)^T = yx^T$.

6. (a) $\|A \otimes B\|_2 = \sigma_1(A \otimes B)$. By theorem 13.10, $\sigma_1(A \otimes B)$ is $\sigma_1(A)\sigma_1(B)$. So $\|A \otimes B\|_2 = \|A\|_2\|B\|_2$.

(b) \[\|A \otimes B\|_F^2 = \text{tr } (A \otimes B)^T (A \otimes B) \quad \text{definition}
\]

\[= \text{tr } (A^T \otimes B^T)(A \otimes B) \quad \text{Theorem 13.4}
\]

\[= \text{tr } (A^T A \otimes B^T B) \quad \text{Theorem 13.3}
\]

\[= (\text{tr } A^TA)(\text{tr } B^TB) \quad \text{Corollary 13.13(1)}
\]

\[= \|A\|_F^2\|B\|_F^2
\]

So, $\|A \otimes B\|_F = \|A\|_F\|B\|_F$. 1
7. (a) By induction with Theorem 13.3.

(b) 
\[ e^{I \otimes A} = \sum_{k=0}^{\infty} \frac{(I \otimes A)^k}{k!} = \sum_{k=0}^{\infty} \frac{I \otimes A^k}{k!} \] by (a) 
\[ = I \otimes \sum_{k=0}^{\infty} \frac{A^k}{k!} = I \otimes e^A \]

Similar \( e^{B \otimes I} = e^B \otimes I \).

(c) \((I \otimes A)(B \otimes I) = B \otimes A\) by Theorem 13.3 and \((B \otimes I)(I \otimes A) = B \otimes A\) by Theorem 13.3.
So \(I \otimes A\) and \(B \otimes I\) commute.

(d) \(e^{A \oplus B} = e^{(I \otimes A + B \otimes I)} = e^{I \otimes A} e^{B \otimes I} = e^A \otimes e^B\) by (c) and p.110 #4
or \(e^{A \oplus B} = e^{(I \otimes A + B \otimes I)} = e^{B \otimes I} e^{I \otimes A} = (e^B \otimes I)(I \otimes e^A) = e^B \otimes e^A\).

Remark: \(A \oplus B\) is not a good notation (p.142).

8. Direct substitution verifies that \(X_s\) and \(X_{ns}\) are solutions. The eigenvalues of \(A\) are ±1 and the eigenvalues of \(-A^T\) are ±1 as well. So the condition in Theorem 13.21 that \(A\) and \(-A^T\) do not have the same eigenvalues is not met.

9. (a) Suppose \(C - XA + DX - XBX = 0\) (Riccati equation). Let \(T = \begin{bmatrix} I & 0 \\ X & I \end{bmatrix}\). Then
\[ T^{-1} = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \] (p.48).

\[ T^{-1}ST = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ X & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix} \begin{bmatrix} A + BX & B \\ C + DX & D \end{bmatrix} = \begin{bmatrix} A + BX & B \\ -XA - XBX + C + DX & -XB + D \end{bmatrix} = \begin{bmatrix} A + BX & B \\ 0 & D - XB \end{bmatrix} \]

(b) \(T\) would then be upper triangular and similar result holds.

10. (a) 
\[ T^{-1}ST = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \Leftrightarrow ST = T \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \]
\[ \Leftrightarrow \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & Y \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \]
\[ \Leftrightarrow AY - YD = -B \quad \text{by comparing the (1,2) block} \]
(b) Find \( R = \begin{bmatrix} I & 0 \\ Y & I \end{bmatrix} \) so that \( R^{-1}SR = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \). Then \( S \) is similar to \( \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \) via \( R \) if and only if \( S^T \) is similar to \( \begin{bmatrix} A^T & 0 \\ 0 & D^T \end{bmatrix} \) via \( T = R^T \). If \( A^TY^T - Y^TD^T = -C^T \), i.e., \( Y \) satisfies the Sylvester equation \( YA - DY = -C \).

Math 5050/6050 Supplementary notes to Chapter 13


**Section 13.2** The proof of Theorem 13.12 is just a basic idea and does not take care of multiplicities of eigenvalues. Better use Jordan canonical form of Schur triangularization form in Chapter 9, alike the remark on Theorem 13.16.

**Section 13.3** The first part of Theorem 13.21 is a Corollary of Theorem 13.18 in which \( B = A^T \). Suppose in addition that \( C \) is symmetric. Let \( X \) be the unique solution to \( AX + XA^T = C \). Then \( X^TA^T + AX = C^T = C \), i.e., \( X^T \) is also a solution to the equation. But the solution is unique so that \( X = X^T \), i.e., \( X \) is symmetric.

Proof of Theorem 13.26: Let \( A \in \mathbb{R}^{m \times n} \), \( B \in \mathbb{R}^{n \times p} \) and \( C \in \mathbb{R}^{p \times q} \). The \( i \)th column of \( AB \) is \( Ab_i \) and the \( j \)th column of \( ABC \) is

\[
ABC_j = \sum_{i=1}^{p} c_{ij} Ab_i = (c_j^T \otimes A) \text{vec } B
\]

so that \( \text{vec } (ABC) = (C^T \otimes A) \text{vec } B \).

On p.148, Use (P1)-(P4) to check \( (M \otimes N)^+ = M^+ \otimes N^+ \).