Math 5050/6050 Key to Chapter 11

Exercises

1. Skip

2. Suppose $x, y \in \mathbb{R}$, $A = xy^T$, $\alpha = x^T y$. Then

$$\begin{aligned}
e^{tA} &= I + tA + \frac{t^2A^2}{2!} + \frac{t^3A^3}{3!} + \cdots \\
&= I + txy^T + \frac{t^2xy^Txy^T}{2!} + \frac{t^3xy^Txy^Txy^T}{3!} + \cdots \\
&= I + x(t + \frac{t^2y^Tx}{2!} + \frac{t^3(y^Tx)^2}{3!} + \cdots)y^T \\
&= I + (t + \frac{\alpha t^2}{2!} + \frac{(\alpha)^2t^3}{3!} + \cdots)xy^T \qquad \text{because } (\cdots) \text{ is a scalar}
\end{aligned}$$

Now

$$\begin{aligned}
g(t,\alpha) &:= t + \frac{\alpha t^2}{2!} + \frac{\alpha^2 t^3}{3!} + \cdots \\
&= \begin{cases} \frac{1}{\alpha}(\alpha t + \frac{\alpha^2t^2}{2!} + \frac{\alpha^3t^3}{3!} + \cdots) & \text{if } \alpha \neq 0 \\ t & \text{if } \alpha = 0 \end{cases} \\
&= \begin{cases} \frac{1}{\alpha}(e^{\alpha t} - 1) & \text{if } \alpha \neq 0 \\ t & \text{if } \alpha = 0 \end{cases}
\end{aligned}$$

3. Let $A = \begin{bmatrix} I & X \\ 0 & -I \end{bmatrix}$, where $X \in \mathbb{R}^{m\times n}$. Then

$$A^2 = \begin{bmatrix} I & X \\ 0 & -I \end{bmatrix}\begin{bmatrix} I & X \\ 0 & -I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

So $A^3 = A$, $A^4 = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \cdots$ Thus

$$\begin{aligned}
e^A &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} I & X \\ 0 & -I \end{bmatrix} + \frac{1}{2!}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} + \frac{1}{3!}\begin{bmatrix} I & X \\ 0 & -I \end{bmatrix} + \cdots \\
&= \begin{bmatrix} I(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots) & X + \frac{X}{3!} + \frac{X}{5!} + \cdots \\ 0 & I(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots) \end{bmatrix} \\
&= \begin{bmatrix} eI & (\sinh 1)X \\ 0 & \frac{I}{e} \end{bmatrix},
\end{aligned}$$

where $\sinh x := \frac{e^x - e^{-x}}{2} = -i\sin ix = x + \frac{x}{3!} + \frac{x}{5!} + \cdots$.

4. (a) Suppose that H is Hamiltonian, i.e. $K^{-1}H^TK = -H$. Since $-H$ and H^T are similar, they have the same set of eigenvalues. But the eigenvalues of $-H$ are the negatives of the eigenvalues of H, and the eigenvalues of H^T are those of H. Hence, if λ is an eigenvalue of H, so is $-\lambda$.

(b) Similar to (a).

(c)

$$\begin{aligned} K^{-1}(S^{-1}HS)^TK &= K^{-1}S^TH^T(S^T)^{-1}K \\ &= K^{-1}S^TK\cdot K^{-1}H^TK\cdot K^{-1}(S^T)^{-1}K \\ &= S^{-1}(-H)(S^{-1})^{-1} \\ &= -S^{-1}(H)S \end{aligned}$$

So, $S^{-1}HS$ is Hamiltonian.

(d)

$$\begin{aligned} K^{-1}(e^H)^TK &= K^{-1}e^{H^T}K \qquad \text{by } (e^H)^T = e^{H^T} \\ &= e^{K^{-1}H^TK} \qquad \text{by } K^{-1}e^AK = e^{K^{-1}AK} \\ &= e^{-H} \qquad \text{since } H \text{ is Hamiltonian} \\ &= (e^H)^{-1}. \end{aligned}$$

So e^H is symplectic.

5. Define a map $\varphi : \mathbb{C} \to R := \left\{ \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} : \alpha, \beta \in \mathbb{R} \right\}$ by

$$\varphi(\alpha + i\beta) := \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Clearly $\varphi(1) = I$ and φ is a linear transformation where $\mathbb{C}$ and R are viewed as vector spaces over $\mathbb{R}$.

Claim: $\varphi(\xi_1\xi_2) = \varphi(\xi_1)\varphi(\xi_2)$ for all $\xi_1, \xi_2 \in \mathbb{C}$.

Let $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. Then

$$\begin{aligned} e^{tA} &= e^{t\varphi(\alpha+i\beta)} \\ &= I + t\varphi(\alpha+i\beta) + \frac{t^2}{2!}(\varphi(\alpha+i\beta))^2 + \cdots \\ &= I + t\varphi(\alpha+i\beta) + \frac{t^2}{2!}\varphi[(\alpha+i\beta)^2] + \cdots \quad \text{since } \varphi(\xi_1\xi_2) = \varphi(\xi_1)\varphi(\xi_2) \\ &= \varphi(1 + t(\alpha+i\beta) + \frac{t^2}{2!}(\alpha+i\beta)^2 + \cdots) \\ &= \varphi(e^{t(\alpha+i\beta)}) \\ &= \varphi(e^{t\alpha}e^{it\beta}) \\ &= \varphi(e^{t\alpha})\varphi(e^{it\beta}) \\ &= \varphi(e^{t\alpha})\varphi(\cos t\beta + i\sin t\beta) \\ &= \begin{bmatrix} e^{t\alpha} & 0 \\ 0 & e^{t\alpha} \end{bmatrix} \begin{bmatrix} \cos t\beta & \sin t\beta \\ -\sin t\beta & \cos t\beta \end{bmatrix} \\ &= \begin{bmatrix} e^{t\alpha}\cos t\beta & e^{t\alpha}\sin t\beta \\ -e^{t\alpha}\sin t\beta & e^{t\alpha}\cos t\beta \end{bmatrix} \end{aligned}$$

6. This exercise should be placed before Exercise 5 and would lead to a solution to Exercise 5. Let $A = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$. By direct calculation, the eigenvalues are $\alpha \pm i\beta$. So A is always diagonalizable (when $\beta \neq 0$, distinct eigenvalues; when $\beta = 0$, $A = I$). Indeed the matrix is normal and thus is unitarily diagonalizable.

Case 1: $\lambda = \alpha + i\beta$ leads to the unit eigenvector $x = \frac{1}{\sqrt{2}} \begin{bmatrix} -i \\ 1 \end{bmatrix}$.

Case 2: $\lambda = \alpha - i\beta$ leads to the unit eigenvector $x = \frac{1}{\sqrt{2}} \begin{bmatrix} i \\ 1 \end{bmatrix}$.

Let $U = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}$ which is unitary. Then $U^H A U = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix}$. Thus

$$(U^H A U)^k = \begin{bmatrix} \alpha + i\beta & 0 \\ 0 & \alpha - i\beta \end{bmatrix}^k$$

so that

$$\begin{aligned} A^k &= U \begin{bmatrix} (\alpha + i\beta)^k & 0 \\ 0 & (\alpha - i\beta)^k \end{bmatrix} U^H \\ &= \frac{1}{2} \begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (\alpha + i\beta)^k & 0 \\ 0 & (\alpha - i\beta)^k \end{bmatrix} \begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} (\alpha + i\beta)^k + (\alpha - i\beta)^k & -i((\alpha + i\beta)^k - (\alpha - i\beta)^k) \\ i((\alpha + i\beta)^k - (\alpha - i\beta)^k) & (\alpha + i\beta)^k + (\alpha - i\beta)^k \end{bmatrix} \end{aligned}$$

Another way to do Exercise 5:

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} t^k A^k / k! \\ &= \frac{1}{2} \sum_{k=0}^{\infty} t^k \begin{bmatrix} (\alpha + i\beta)^k + (\alpha - i\beta)^k & -i((\alpha + i\beta)^k - (\alpha - i\beta)^k) \\ i((\alpha + i\beta)^k - (\alpha - i\beta)^k) & (\alpha + i\beta)^k + (\alpha - i\beta)^k \end{bmatrix} / k! \\ &= \frac{1}{2} \begin{bmatrix} \sum_{k=0}^{\infty} t^k (\alpha + i\beta)^k / k! + \sum_{k=0}^{\infty} t^k (\alpha - i\beta)^k / k! & -i(\sum_{k=0}^{\infty} t^k (\alpha + i\beta)^k / k! - \sum_{k=0}^{\infty} t^k (\alpha - i\beta)^k / k!) \\ i(\sum_{k=0}^{\infty} t^k (\alpha + i\beta)^k / k! - \sum_{k=0}^{\infty} t^k (\alpha - i\beta)^k / k!) & \sum_{k=0}^{\infty} t^k (\alpha + i\beta)^k / k! + \sum_{k=0}^{\infty} t^k (\alpha - i\beta)^k / k! \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} e^{t(\alpha + i\beta)} + e^{t(\alpha - i\beta)} & -i(e^{t(\alpha + i\beta)} - e^{t(\alpha - i\beta)}) \\ i(e^{t(\alpha + i\beta)} - e^{t(\alpha - i\beta)}) & e^{t(\alpha + i\beta)} + e^{t(\alpha - i\beta)} \end{bmatrix} \\ &= \frac{e^{\alpha t}}{2} \begin{bmatrix} e^{it\beta} + e^{-it\beta} & -i(e^{it\beta} - e^{-it\beta}) \\ i(e^{it\beta)} - e^{-it\beta}) & e^{it\beta} + e^{-it\beta} \end{bmatrix} \\ &= \begin{bmatrix} e^{t\alpha} \cos t\beta & e^{t\alpha} \sin t\beta \\ -e^{t\alpha} \sin t\beta & e^{t\alpha} \cos t\beta \end{bmatrix}. \end{aligned}$$

7. (a) By Exercise 5, $e^{tA} = e^{t\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}} = \begin{bmatrix} e^{2t} \cos t & e^{2t} \sin t \\ -e^{2t} \sin t & e^{2t} \cos t \end{bmatrix}$. The other two parts are similar.

8. (a) By Theorem 11.2,

$$\begin{aligned} x(t) &= e^{(t-t_0)A}x_0 \\ &= e^{tA}x_0 \qquad \text{since } t_0 = 0 \\ &= e^{t\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}}\begin{bmatrix}1\\2\end{bmatrix}. \end{aligned}$$

$$A = \begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}, A^2 = \begin{bmatrix}1 & -2\\ 0 & 1\end{bmatrix}, A^3 = \begin{bmatrix}-1 & 3\\ 0 & -1\end{bmatrix}, A^4 = \begin{bmatrix}1 & -4\\ 0 & 1\end{bmatrix}, \cdots, A^n = \begin{bmatrix}(-1)^n & (-1)^{n-1}n\\ 0 & (-1)^n\end{bmatrix}.$$

Or

$$\begin{aligned} A^n &= \left(\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix} + \begin{bmatrix}0 & 1\\ & 0\end{bmatrix}\right)^n \\ &= \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}^n + \binom{n}{1}\begin{bmatrix}-1 & \\ & -1\end{bmatrix}^{n-1}\begin{bmatrix}0 & 1\\ & 0\end{bmatrix} + \binom{n}{2}\begin{bmatrix}-1 & \\ & -1\end{bmatrix}^{n-2}\begin{bmatrix}0 & 1\\ & 0\end{bmatrix}^2 + \cdots + \begin{bmatrix}0 & 1\\ & 0\end{bmatrix}^n \\ &= \begin{bmatrix}(-1)^n & 0\\ 0 & (-1)^n\end{bmatrix} + (-1)^{n-1}n\begin{bmatrix}0 & 1\\ & 0\end{bmatrix} \qquad \text{since } \begin{bmatrix}0 & 1\\ & 0\end{bmatrix}^2 = 0 \end{aligned}$$

Thus

$$\begin{aligned} x(t) &= e^{t\begin{bmatrix}-1 & 1\\ 0 & -1\end{bmatrix}}\begin{bmatrix}1\\2\end{bmatrix} \\ &= (\sum_{n=0}^{\infty}\frac{1}{n!}t^n\begin{bmatrix}(-1)^n & (-1)^{n-1}n\\ 0 & (-1)^n\end{bmatrix})\begin{bmatrix}1\\2\end{bmatrix} \\ &= (\sum_{n=0}^{\infty}\frac{1}{n!}\begin{bmatrix}t^n(-1)^n & (-1)^{n-1}nt^n\\ 0 & t^n(-1)^n\end{bmatrix})\begin{bmatrix}1\\2\end{bmatrix} \\ &= \begin{bmatrix}e^{-t} & te^{-t}\\ 0 & e^{-t}\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} \\ &= \begin{bmatrix}e^{-t} + 2te^{-t}\\ 2e^{-t}\end{bmatrix} \end{aligned}$$

9. By Theorem 11.2, $x(t) = e^{tA}x_0$, where A is skew symmetric. Notice that e^{tA} is orthogonal since

$$\begin{aligned} (e^{tA})(e^{tA})^T &= e^{tA}e^{tA^T} \qquad \text{by property 2 on p.109} \\ &= e^{tA}e^{-tA} \\ &= 0 \qquad \text{by property 5} \end{aligned}$$

So $||x(t)||_2 = ||e^{tA}x_0||_2 = ||x_0||_2$.

10. By Theorem 11.6, $X(t) = e^{tA}Ce^{-tA}$, which is similar to C. So the eigenvalues of C and those of $X(t)$ are the same.

11. Skip

12. Skip

Math 5050/6050 Supplementary notes to Chapter 11

Section 11.1.2 Proof of Theorem 11.2: Differentiate $x(t) = e^{(t-t_0)A}x_0$ with respect to t and use property 7 to have

$$\dot{x}(t) = Ae^{(t-t_0)A}x_0 = Ax(t).$$

When $t = t_0$, $x(t_0) = e^{(t_0-t_0)A}x_0 = e^0 x_0 = x_0$ since $e^0 = I$ by property 1.

Section 11.1.4 Proof of Theorem 11.5 is identical to that of Theorem 11.2: Differentiate $X(t) = e^{(t-t_0)A}C$ with respect to t and use property 7 to have

$$\dot{X}(t) = Ae^{(t-t_0)A}C = AX(t).$$

When $t = t_0$, $X(t_0) = e^{(t_0-t_0)A}C = e^0 C = C$ since $e^0 = I$ by property 1.

Or write $C = [c_1 \cdots c_n]$ and $X = [x_1 \cdots x_n]$ and use Theorem 11.2.

Remark: However $Y(t) = Ce^{(t-t_0)A}$ is not a solution since $\dot{Y}(t) = CAe^{(t-t_0)A} \neq AY(t)$ in general unless A and C commute.

Proof of Theorem 11.5: Differentiate $X(t) = e^{tA}Ce^{tB}$ with respect to t and use property 7 to have

$$\dot{X}(t) = Ae^{tA}Ce^{tB} + e^{tA}Ce^{tB}B = AX(t) + X(t)B.$$

When $t = t_0$, $X(t_0) = e^0 Ce^0 = C$ since $e^0 = I$ by property 1. When $B = A^T$ we have Corollary 11.7.