Math 5050/6050 Key to Chapter 1

Exercises

1. $\det(\alpha A) = \alpha^n \det A$ by Property 5 on p.5. So $\det(-A) = (-1)^n \det A$.

2. A orthogonal means that $A^T A = I = AA^T$. Taking determinant on both sides and using Property 9 and Property 14 yield

$$(\det A)^2 = \det A^T \det A = 1$$

since $\det I = 1$. Hence $\det A = \pm 1$.

Example:

(a) $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is orthogonal (Check!) and $\det A = \cos^2\theta + \sin^2\theta = 1$.

(b) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (check!) and $\det A = -1$.

A unitary means that $A^H A = I = AA^H$. With the same reasoning $|\det A| = 1$ (Check!)

3. Let $x, y \in \mathbb{R}^n$. Then

$$\begin{bmatrix} 1 & y^T \\ x & I \end{bmatrix} \begin{bmatrix} 1 & -y^T \\ 0 & I \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ x & I - xy^T \end{bmatrix}.$$

If we switch the two matrices:

$$\begin{bmatrix} 1 & -y^T \\ 0 & I \end{bmatrix} \begin{bmatrix} 1 & y^T \\ x & I \end{bmatrix} = \begin{bmatrix} 1 - y^T x & 0 \\ x & I \end{bmatrix}.$$

Since $\det(AB) = \det(BA)$, we have $\det(I - xy^T) = 1 - y^T x$.

4. $(U_1 U_2 \cdots U_k)^T (U_1 U_2 \cdots U_k) = U_k^T \cdots U_2^T U_1^T U_1 U_2 \cdots U_k = I$ since $U_1, \ldots, U_k$ are orthogonal. Indeed it suffices to show the case $k = 2$. The general case k follows then.

Question: How about unitary matrices?

5. (a) The diagonal entries of $\alpha A + \beta B$ are $\alpha a_{ii} + \beta b_{ii}$. So

$$\mathrm{tr}\,(\alpha A + \beta B) = \sum_{i=1}^n (\alpha a_{ii} + \beta b_{ii}) = \sum_{i=1}^n \alpha a_{ii} + \sum_{i=1}^n \beta b_{ii} = \alpha \sum_{i=1}^n a_{ii} + \beta \sum_{i=1}^n b_{ii} = \alpha \mathrm{tr}\, A + \beta \mathrm{tr}\, B.$$

In other words, tr is linear.

(b) The diagonal entries of AB are $\sum_{k=1}^n a_{ik} b_{ki}$. So

$$\mathrm{tr}\,(AB) = \sum_{i=1}^n (\sum_{k=1}^n a_{ik} b_{ki}) = \sum_{k=1}^n (\sum_{i=1}^n a_{ik} b_{ki}) = \sum_{k=1}^n (\sum_{i=1}^n b_{ki} a_{ik}) = \mathrm{tr}\,(BA).$$

It is a very nice property of tr.

(c) If S is real skew symmetric, i.e., $S^T = -S$, then the diagonal entries are all zero (since $s_{ii} = -s_{ii}$). So $\mathrm{tr}\, S = 0$. The converse is not true, for example, $S = \begin{bmatrix} 1 & 0 \\ 3 & -1 \end{bmatrix}$ so that $\mathrm{tr}\, S = 0$ but S is not skew symmetric.

6. (a)

$$A = \frac{1}{2}\begin{bmatrix} 2\cos^2\theta & \sin 2\theta \\ \sin 2\theta & 2\sin^2\theta \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1+\cos 2\theta & \sin 2\theta \\ \sin 2\theta & 1-\cos 2\theta \end{bmatrix} = \frac{1}{2}\left(I + \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}\right).$$

So

$$\begin{aligned} A^2 &= \frac{1}{4}\left(I + 2\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} + \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}^2\right) \\ &= \frac{1}{4}\left(I + 2\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} + I\right) = A \end{aligned}$$

(Check!)

(b) Use contrapositive argument: Given $A^2 = A$, if A is nonsingular, i.e., A^{-1} exists, then $A^{-1}A^2 = A^{-1}A$, i.e., $A = I$.

Note: If you use det argument: $A^2 = A$ implies that $(\det A)^2 = \det A$, then $\det A = 0$ or 1 which does not necessarily imply $A = I$. For example $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\det A = 0$ (A is singular).