# Operator properties of $T$ and $K(T)$

Chi-Kwong Li*
Department of Mathematics
College of William and Mary
P.O.Box 8795, Williamsburg
Virginia 23187-8795, USA
E-mail: ckli@math.wm.edu
and
Tin-Yau Tam
Department of Mathematics
Auburn University
Auburn, Alabama 36849-5310, USA
E-mail: tamtiny@auburn.edu

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**Dedicated to Professor Graciano de Oliveira on the occasion of his retirement.**

**Abstract**

Let $V$ be an $n$-dimensional inner product space over $\mathbb{C}$, and let $H$ be a subgroup of the symmetric group on $\{1, \ldots, m\}$. Suppose $\chi : H \to \mathbb{C}$ is an irreducible character (not necessarily of degree 1). Let $V_\chi^m(H)$ denote the symmetry class of tensors over $V$ associated with $H$ and $\chi$ and let $K(T) \in \mathrm{End}\,(V_\chi^m(H))$ be the induced operator of $T \in \mathrm{End}\,(V)$.

It is known that if $T$ is normal, unitary, positive (semi-)definite, Hermitian, then $K(T)$ has the corresponding property. Furthermore, if $T_1 = \xi T_2$ for some $\xi \in \mathbb{C}$ with $\xi^m = 1$, then $K(T_1) = K(T_2)$. The converse of these statements are not valid in general. Necessary and sufficient conditions on $\chi$ and the operators $T, T_1, T_2$ ensuring the validity of the converses of the above statements are given. These extend the results of those on linear characters by Li and Zaharia.

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## 1 Introduction

Let $V$ be an $n$-dimensional inner product space over $\mathbb{C}$. Let $S_m$ be the symmetric group of degree $m$ on the set $\{1, \ldots, m\}$. Each $\sigma \in S_m$ gives rise to a linear operator $P(\sigma)$ on $\otimes^m V$:

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}, \qquad v_1, \ldots, v_m \in V$$

on the *decomposable tensors* $v_1 \otimes v_2 \otimes \cdots \otimes v_m$.

Suppose $H$ is a subgroup of $S_m$, and $\chi : H \to \mathbb{C}$ is an irreducible character of $H$ (not necessarily linear). The *symmetrizer*

$$S_\chi := \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma) \in \operatorname{End}(\otimes^m V)$$

is an orthoprojector with respect to the induced inner product on $\otimes^m V$:

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} (u_i, v_i),$$

and the range of $S_\chi$

$$V_\chi^m(H) := S_\chi(\otimes^m V)$$

is called the *symmetry class of tensors* over $V$ associated with $H$ and $\chi$. The elements in $V_\chi^m(H)$ of the form $S_\chi(v_1 \otimes \cdots \otimes v_m)$ are called *decomposable symmetrized tensors* and are denoted by $v_1 * \cdots * v_m$.

For any $T \in \operatorname{End}(V)$, there is a unique induced operator $K(T)$ acting on $V_\chi^m(H)$ satisfying

$$K(T) v_1 * \ldots * v_m = T v_1 * \cdots * T v_m.$$

Indeed $V_\chi^m(H)$ is stable under $\otimes^m T$ and $K(T) = \otimes^m T|_{V_\chi^m(H)}$. Thus $K(T)v^* = (\otimes^m T)v^*$, $v^* \in V_\chi^m(H)$. Clearly $K(\xi T) = \xi^m K(T)$, $\xi \in \mathbb{C}$.

It is known (see [8]) that if $T$ is normal, unitary, positive (semi-)definite, and (skew) Hermitian, then $K(T)$ has the corresponding property; if $T_1 = \xi T_2$ for some complex number $\xi$ with $\xi^m = 1$, then $K(T_1) = K(T_2)$. But the converses are not true in general. For linear characters $\chi$, Li and Zaharia [7] gave necessary and sufficient conditions on $\chi$ and the operators $T$, $S$ so that the following hold.

(I) If $K(T) \neq 0$ is normal or unitary, then $T$ has the corresponding property.

(II) If there exists $\eta \in \mathbb{C}$ with $|\eta| = 1$ such that $\eta K(T) \neq 0$ is Hermitian (respectively, positive definite or positive semi-definite), then $\xi T$ also has the corresponding property for some $\xi \in \mathbb{C}$ with (i) $\xi^m = \eta$ or (ii) $m$ is even $\xi^m = -1$ (respectively, $\xi^m = 1$).

(III) Suppose $K(T) \neq 0$. Then a linear operator $S$ satisfies $K(S) = K(T)$ if and only if $S = \xi T$ for some $\xi \in \mathbb{C}$ with $\xi^m = 1$.

The results in [7] explained all the known counterexamples and existing results in the literature. The purpose of this paper is to extend the results in [7] to nonlinear irreducible characters. The structure of symmetry classes of tensors and induced operators associated with nonlinear characters are more complicated than that corresponding to the linear characters. For example, if $\chi$ is linear and if $T$ has a matrix representation with respect to an orthonormal basis, then $K(T)$ has a natural matrix representation in terms of a corresponding orthonormal basis. But this is not true for nonlinear irreducible characters (see the next section). In [7], the analysis depends heavily on the matrix representation of $T$ and $K(T)$ with respect to the standard orthonormal bases. It is not clear from the proofs in [7] that the results are also valid for nonlinear characters as well. To get around the problem mentioned above, we do not fix matrix representations in advance. The key steps in our proofs often involve choosing triangular bases for $T$ and $K(T)$ judiciously. Once the suitable bases for $T$ and $K(T)$ are chosen, some arguments are quite similar to those in [7]. Nonetheless, for the sake of completeness and easy reference, we choose to include those arguments.

Our paper is organized as follows. In Section 2, we give some preliminary results for induced operators. In Section 3 we present several lemmas. In Section 4 we divide $(\chi, n)$ into several classes, that will determine whether (I), (II), or (III) hold subsequently; some examples will be given to these classes. In Section 5 we determine the necessary and sufficient conditions on the irreducible character $\chi$ on $H \leq S_m$ and operators $T$ and $S$ on $V$ for which (I) or (II) holds. In Section 6 we determine when (III) holds.

## 2 Preliminaries

In this section, we present some preliminary results for induced operators. One may see [8, 9, 11, 12] for some general background.

Let $I(H)$ be the set of irreducible characters of $H$. If $\chi, \xi \in I(H)$ and $\chi \neq \xi$, then $S_\chi S_\xi = 0$. Moreover $\sum_{\chi \in I(H)} S_\chi$ is the identity operator on $\otimes^m V$. So we have the orthogonal sum

$$\otimes^m V = \dot{\sum}_{\chi \in I(G)} V_\chi^m(H).$$

Let $\Gamma_{m,n}$ be the set of sequences $\alpha = (\alpha(1), \ldots, \alpha(m))$ with $1 \leq \alpha(j) \leq n$ for $j = 1, \ldots, m$. Two sequences $\alpha$ and $\beta$ in $\Gamma_{m,n}$ are said to be equivalent modulo $H$, denoted by $\alpha \sim \beta$, if there exists $\sigma \in H$ such that $\beta = \alpha\sigma$. This equivalence relation partitions $\Gamma_{m,n}$ into equivalence classes. Let $\Delta$ be a system of representatives for the equivalence classes so that each sequence in $\Delta$ is first in lexicographic order in its equivalence class. Define $\bar{\Delta}$ as the subset of $\Delta$ consisting of those sequences $\alpha \in \Delta$ such that

$$\nu_\alpha := \sum_{\sigma \in H_\alpha} \chi(\sigma) \neq 0,$$

where $H_\alpha := \{\sigma \in H : \alpha\sigma = \alpha\}$ is the stabilizer subgroup of $\alpha$, that is, $(\chi, 1)_{H_\alpha} := \frac{1}{|H_\alpha|} \sum_{\sigma \in H_\alpha} \chi(\sigma) \neq 0$, or equivalently, the restriction of $\chi$ to $H_\alpha$ contains the principal character as an irreducible constituent. Indeed $(\chi, 1)_{H_\alpha} \neq 0$ amounts to $(\chi, 1)_{H_\alpha} > 0$ since $(\chi, 1)_{H_\alpha}$ is the number of occurrences of the principal character in the restriction of $\chi$ to $H_\alpha$.

If $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis for $V$, then $\{e_\alpha^\otimes := e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,n}\}$ is a basis for $\otimes^m V$. Let

$$e_\alpha^* := S_\chi e_\alpha^\otimes = \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) e_{\alpha\sigma^{-1}(1)} \otimes \cdots \otimes e_{\alpha\sigma^{-1}(m)},$$

for each $\alpha \in \Gamma_{m,n}$. Then $\{e_\alpha^* : \alpha \in \Gamma_{m,n}\}$ is a spanning set for the space $V_\chi^m(H)$, but it may not be linearly independent. Indeed some vectors may even be zero. It can be shown that $e_\alpha^* \neq 0$ if and only if the restriction of $\chi$ to $H_\alpha$ contains the principal character as an irreducible constituent. Let

$$\Omega := \{\alpha \in \Gamma_{m,n} : (\chi, 1)_{H_\alpha} > 0\},$$

and hence $\bar{\Delta} = \Delta \cap \Omega$. For any $\tau \in S_n$, $H_{\tau\alpha} = H_\alpha$ and thus

$$\tau\Omega = \Omega, \qquad \tau \in S_n. \tag{2.1}$$

(Note that for $\alpha \in \Gamma_{m,n}$ we write $\alpha\sigma$ for $\sigma \in S_m$ permuting the entries of $(\alpha(1), \ldots, \alpha(m))$, and we write $\tau\alpha = (\tau(\alpha(1)), \ldots, \tau(\alpha(m)))$ for $\tau \in S_n$ that changes the entries of $\alpha$.) The set $\{e_\alpha^* : \alpha \in \Omega\}$ consists of the nonzero elements of $\{e_\alpha^* : \alpha \in \Gamma_{m,n}\}$. Moreover

$$V_\chi^m(H) = \oplus_{\alpha \in \bar{\Delta}} \langle e_{\alpha\sigma}^* : \sigma \in H \rangle, \tag{2.2}$$

a direct sum of the *orbital subspaces* $O_\alpha := \langle e_{\alpha\sigma}^* : \sigma \in H \rangle$, $\alpha \in \bar{\Delta}$, which denotes the span of the set $\{e_{\alpha\sigma}^* : \sigma \in H\}$ and Freese's theorem asserts that

$$\dim O_\alpha = s_\alpha := \frac{\chi(e)}{|H_\alpha|} \sum_{\sigma \in H_\alpha} \chi(\sigma) = \chi(e)(\chi, 1)_{H_\alpha}. \tag{2.3}$$

Thus the set $\mathcal{B}^* := \{e_\alpha^* : \alpha \in \bar{\Delta}\}$ is a linearly independent set. We now construct a basis for $V_\chi^m(H)$. For each $\alpha \in \bar{\Delta}$, we find a basis for the orbital subspace $O_\alpha$: choose a set

$\{\alpha_1, \ldots, \alpha_{s_\alpha}\}$ from $\{\alpha\sigma : \sigma \in H\}$ such that $\{e^*_{\alpha_1}, \ldots, e^*_{\alpha_{s_\alpha}}\}$ is a basis for $O_\alpha$. Execute this procedure for each $\gamma \in \bar{\Delta}$. If $\{\alpha, \beta, \cdots\}$ is the lexicographically ordered set $\bar{\Delta}$, take

$$\hat{\Delta} = \{\alpha_1, \ldots, \alpha_{s_\alpha}, \beta_1, \ldots, \beta_{s_\beta}, \ldots\}$$

to be ordered as indicated. The elements of $\hat{\Delta}$ are no longer lexicographically ordered and $\hat{\mathcal{B}}^* := \{e^*_\alpha : \alpha \in \hat{\Delta}\}$ is a basis for $V^m_\chi(H)$. Clearly $\bar{\Delta} \subset \hat{\Delta} \subset \Omega$. Though $\hat{\Delta}$ is not unique, it does not depend on the basis $\mathcal{B}$ since $\Delta$ and $\overline{\Delta}$ do not depend on $\mathcal{B}$. Thus if $\mathcal{B}' = \{f_1, \ldots, f_n\}$ is another basis for $V$, then $\{f^*_\alpha : \alpha \in \hat{\Delta}\}$ is still a basis for $V^m_\chi(H)$.

The inner product of $V$ induces an inner product on $V^m_\chi(H)$:

$$(u^*, v^*) = \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) \prod_{t=1}^{m} (u_t, v_{\sigma(t)}). \tag{2.4}$$

So if $\mathcal{B} = \{e_1, \ldots, e_n\}$ is an orthonormal basis for $V$, then

$$(e^*_\alpha, e^*_\beta) = \begin{cases} 0 & \text{if } \alpha \not\sim \beta \\ \frac{\chi(e)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma) & \text{if } \alpha = \beta, \end{cases}$$

and thus

$$\|e^*_\alpha\|^2 = \frac{\chi(e)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma).$$

Hence (2.2) becomes $V^m_\chi(H) = \dot{\sum}_{\alpha \in \bar{\Delta}} \langle e^*_{\alpha\sigma} : \sigma \in H \rangle$, an orthogonal sum. However, those $e^*_\alpha$'s of $\{e^*_\alpha : \alpha \in \hat{\Delta}\}$ belonging to the same orbital subspace need not be orthogonal.

It is known [12, p.103] and also follows from (2.3) that $\bar{\Delta} = \hat{\Delta}$ if and only if $\chi$ is linear. In such cases, $\{e^*_\alpha : \alpha \in \bar{\Delta}\}$ is an orthogonal basis for $V^m_\chi(H)$.

We give several common examples of symmetry classes of tensors and induced operators.

**Example 2.1** Let $1 \leq m \leq n$, $H = S_m$ and $\chi$ be the alternate character, that is, $\chi(\sigma) = \operatorname{sgn}(\sigma)$. Then $V^m_\chi(H)$ is the $m$th exterior space $\wedge^m V$, $\bar{\Delta} = \hat{\Delta} = Q_{m,n}$, the set of strictly increasing sequences in $\Gamma_{m,n}$, $\Delta = G_{m,n}$, the set of nondecreasing sequences in $\Gamma_{m,n}$ and $K(T)$ is the $m$th compound of $T \in \operatorname{End}(V)$, usually denoted by $C_m(T)$.

**Example 2.2** Let $H = S_m$ and $\chi \equiv 1$ be the principal character. Then $V^m_\chi(H)$ is the $m$th completely symmetric space over $V = \mathbb{C}^n$, $\bar{\Delta} = \hat{\Delta} = \Delta = G_{m,n}$, and $K(T)$ is the $m$th induced power of $T \in \operatorname{End}(V)$, usually denoted by $P_m(T)$.

**Example 2.3** Let $H = \{e\}$ where $e$ is the identity in $S_m$ ($\chi \equiv 1$ is the only irreducible character). Then $V^m_\chi(H) = \otimes^m V$, $\bar{\Delta} = \hat{\Delta} = \Delta = \Gamma_{m,n}$, and $K(T) = \otimes^m T$ is the $m$th tensor power of $T \in \operatorname{End}(V)$.

We now provide an example with nonlinear irreducible character.

**Example 2.4** Consider $S_3$ and use the (only) nonlinear irreducible character $\chi = \chi_3$ in [3, p.157], that is, $\chi_3(e) = 2$, $\chi_3((12)) = 0$, $\chi((123)) = -1$. If $n = \dim V = 2$, then [11, p.164]

$$\bar{\Delta} = \{(1,1,2),(1,2,2)\}, \quad \hat{\Delta} = \{(1,1,2),(1,2,1),(1,2,2),(2,1,2)\}.$$

Let $\mathcal{B} = \{e_1, e_2\}$ be a basis for $V$ and let $T \in \mathrm{End}\,(V)$ be defined by

$$[T]_{\mathcal{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\mathcal{B}^* = \{e^*_{(1,1,2)}, e^*_{(1,2,1)}, e^*_{(1,2,2)}, e^*_{(2,1,2)}\}$ is a basis for $V^m_\chi(H)$, and (see [12, p.98-101])

$$e^*_{(2,1,1)} = -e^*_{(1,1,2)} - e^*_{(1,2,1)}, \quad e^*_{(2,2,1)} = -e^*_{(1,2,2)} - e^*_{(2,1,2)}.$$

By direct computation

$$[K(T)]_{\mathcal{B}^*} = \begin{pmatrix} a^2d - abc & 0 & abd - b^2c & 0 \\ 0 & a^2d - abc & abd - b^2c & b^2c - abd \\ acd - bc^2 & 0 & ad^2 - bcd & 0 \\ acd - bc^2 & bc^2 - acd & 0 & ad^2 - bcd \end{pmatrix}$$

Observe that $\mathcal{B}^*$ is not an orthogonal basis even if $\mathcal{B}$ is an orthonormal basis, since

$$(e^*_{(1,1,2)}, e^*_{(1,2,1)}) = (e^*_{(1,2,2)}, e^*_{(2,1,2)}) = -\frac{1}{3}.$$

Let $m_j(\alpha)$ denote the number of occurrence of $j$ in the sequence $\alpha \in \hat{\Delta}$. The following contains some properties of the induced operator.

**Proposition 2.5** [11, 12] *Let $S, T$ be linear operators on $V$ and assume $\bar{\Delta} \neq \phi$.*

(a) $K(I_V) = I_{V^m_\chi(H)}$.

(b) $K(ST) = K(S)K(T)$.

(c) *$T$ is invertible if and only if $K(T)$ is. Moreover, we have $K(T^{-1}) = K(T)^{-1}$.*

(d) $K(T^*) = K(T)^*$.

(e) *If the matrix representation of $T$ with respect to the basis $\mathcal{B}$ is in (lower or upper) triangular or in diagonal form, then so is $K(T)$ with respect to the basis $\hat{\mathcal{B}}^*$.*

(f) *If $T$ is normal, unitary, positive (semi-)definite, Hermitian or skew-Hermitian (when $m$ is odd), so is $K(T)$.*

(g) *If $T$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, and singular values $s_1 \geq \cdots \geq s_n$, then $K(T)$ has eigenvalues $\prod_{j=1}^n \lambda_j^{m_j(\alpha)}$ and singular values $\prod_{j=1}^n s_j^{m_j(\alpha)}$, $\alpha \in \hat{\Delta}$.*

(h) *If* $\mathrm{rank}\,(T) = r$, *then* $\mathrm{rank}\, K(T) = |\Gamma_{m,r} \cap \hat{\Delta}|$.

**Remark 2.6** To prove (g) one may use Schur Triangularization Theorem [11, p.239] to find a triangular basis for $T$ so that the matrix representation of $T$ has diagonal entries $\lambda_1, \ldots, \lambda_n$. In fact, Schur Triangularization Theorem allows any order of $\lambda$'s. Hence, we see that if $(k_1, \ldots, k_n)$ is a sequence of nonnegative integers such that $\prod_{j=1}^n \lambda_j^{k_j}$ is an eigenvalue of $K(T)$ with multiplicity $r$, then for any $\sigma \in S_n$, $\prod_{j=1}^n \lambda_{\sigma(j)}^{k_j}$ is also an eigenvalue of $K(T)$ with multiplicity $r$. As a result, $\operatorname{Tr} K(T)$ is a symmetric polynomial of $\lambda_1, \ldots, \lambda_n$. For the singular values of $T$, we can write $T = U|T|$ for some unitary operators $U$ and a positive semi-definite operator $|T|$ such that $|T|^2 = T^*T$. Then the singular values of $T$ are the eigenvalues of $|T|$, and the singular values of $K(T) = K(U)K(|T|)$ are the eigenvalues of $K(|T|)$. Thus, the assertion follows. We will use condition (g) frequently in our study. This observation on eigenvalues can also be deduced as follows. Since $H_\alpha = H_{\tau\alpha}$, $\tau \in S_n$, $\alpha \in \Gamma_{m,n}$, we have $\alpha \in \bar{\Delta}$ if and only if $\tau\alpha \sim \beta \in \bar{\Delta}$; in addition $\dim O_\alpha = \dim O_\beta$ since $H_{\alpha\sigma} = \sigma^{-1} H_\alpha \sigma$, $\sigma \in H$. Moreover for $\alpha, \alpha' \in \Gamma_{m,n}$, $\tau \in S_n$, $\tau\alpha \sim \tau\alpha'$ if and only if $\alpha \sim \alpha'$. Clearly $m_i(\tau\alpha) = m_{\tau^{-1}(i)}(\alpha)$, $i = 1, \ldots, n$. Denote $m_\alpha := (m_1(\alpha), \ldots, m_n(\alpha))$. Then for any given $\tau \in S_n$, the sets $\{\alpha \in \hat{\Delta} : m_\alpha = (k_1, \ldots, k_n)\}$ and $\{\alpha \in \hat{\Delta} : m_\alpha = (k_{\tau(1)}, \ldots, k_{\tau(n)})\}$ are of the same size where $k$'s are nonnegative integers.

In the subsequent discussion, we shall use $\mu(\bar{\Delta})$ to denote the smallest integer $r$ such that $\Gamma_{m,r} \cap \bar{\Delta} \neq \emptyset$. Similarly we can define $\mu(\hat{\Delta})$ but it is clear that $\mu(\bar{\Delta}) = \mu(\hat{\Delta})$. As a result, an operator $T$ on $V$ satisfies $K(T) = 0$ if and only if $\operatorname{rank}(T) < \mu(\bar{\Delta})$ by Proposition 2.5 (h). Furthermore, we say that $T$ is the direct sum $T_1 \oplus T_2$ if $V$ has a subspace $V_1$ so that $V_1$ is invariant under both $T$ and $T^*$; $T_1$ is then the restriction of $T$ on $V_1$, and $T_2$ is the restriction of $T$ on the orthogonal complement of $V_1$ in $V$. As usual, if $T$ has the matrix representation $A$, then $\operatorname{Tr} T = \operatorname{Tr} A$ and $\det(T) = \det(A)$.

# 3 Some Lemmas

Given two vectors $x, y \in \mathbb{R}^n$, we say that $x$ is *majorized* by $y$ if the sum of entries of the two vectors are the same, and the sum of the $k$ largest entries of $x$ is not larger than that of $y$ for $k = 1, \ldots, n-1$. We need the following result, that also follows from the corollary in [1]. Given an irreducible character $\chi$ on $H \leq S_m$, define

$$\Omega(\chi, H) := \{\alpha \in \Gamma_{m,n} : (\chi, 1)_{H_\alpha} > 0\},$$

and $m(\Omega(\chi, H)) := \{(m_1(\alpha), \ldots, m_n(\alpha)) : \alpha \in \Omega(\chi, H)\} \subseteq \mathbb{N}^n$ be the collection of vectors of multiplicities of all $\alpha \in \Omega(\chi, H)$.

**Lemma 3.1** *Suppose $\chi$ is an irreducible character on $H \leq S_m$. The following conditions are equivalent for a sequence $(k_1, \ldots, k_n) \in \mathbb{N}^n$.*

(a) $(k_1, \ldots, k_n) \in m(\Omega(\chi, H))$.

(b) *There is an irreducible character* $\psi$ *on* $S_m$ *with* $(\psi, \chi)_H \neq 0$ *such that*

$$(k_1, \ldots, k_n) \in m(\Omega(\psi, S_m)),$$

*equivalently,* $(k_1, \ldots, k_n)$ *is majorized by the vector of partition corresponding to the irreducible character* $\psi$.

*Consequently, if* $\alpha \in \bar{\Delta}$ *and if* $(k_1, \ldots, k_n) \in \mathbb{N}^n$ *is majorized by* $(m_1(\alpha), \ldots, m_n(\alpha))$*, then there exists* $\beta \in \bar{\Delta}$ *such that* $(m_1(\beta), \ldots, m_n(\beta)) = (k_1, \ldots, k_n)$.

**Proof:** The equivalence of the conditions in (b) is due to Merris [10]. Also see [11, Theorem 6.37].

Let $\chi$ be an irreducible character of $H \leq S_m$, and $\tilde{\chi}$ the character on $S_m$ induced by $\chi$. Let $G = (S_m)_\alpha$ be the stabilizer of $\alpha \in \Gamma_{m,n}$ in $S_m$. For every $y \in S_m$, one readily checks that $H_{\alpha y} = y^{-1}Gy \cap H$ is the stabilizer of $\alpha y$ in $H$. Then we have

$$\sum_{\sigma \in G} \tilde{\chi}(\sigma) = \sum_{\sigma \in G} \frac{1}{|H|} \sum_{\substack{y \in S_m \\ y^{-1}\sigma y \in H}} \chi(y^{-1}\sigma y) = \frac{1}{|H|} \sum_{y \in S_m} \sum_{\sigma \in H_{\alpha y}} \chi(\sigma). \tag{3.5}$$

Each term on the right side $\sum_{\sigma \in H_{\alpha y}} \chi(\sigma) = |H_{\alpha y}|(\chi, 1)_{H_{\alpha y}} \geq 0$ since $(\chi, 1)_{H_{\alpha y}}$ is the number of occurrences of the principal character in the restriction of $\chi$ to $H_{\alpha y}$.

(a) $\Longrightarrow$ (b) If $\alpha \in \Omega(\chi, H)$, then $\sum_{\sigma \in H_\alpha} \chi(\sigma) > 0$. So the right side of (3.5) is not smaller than $\frac{1}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma) > 0$. Hence, from the left side of (3.5), one of the irreducible constituents of $\tilde{\chi}$ on $S_m$, say $\psi$, must satisfy $\sum_{\sigma \in G} \psi(\sigma) > 0$, that is, $m(\alpha) \in m(\Omega(\psi, S_m))$. Clearly, we have $(\psi, \chi)_H = (\psi, \tilde{\chi})_{S_m} > 0$ by the Frobenius Reciprocity Theorem.

(b) $\Longrightarrow$ (a) Suppose $\psi$ is an irreducible character on $S_m$ such that $(\psi, \chi)_H \neq 0$, and for some $\alpha \in \Gamma_{m,n}$ with $m(\alpha) := (k_1, \ldots, k_n) \in m(\Omega(\psi, S_m))$, that is, $\sum_{\sigma \in G} \psi(\sigma) > 0$. Then the character $\tilde{\chi}$ on $S_m$ induced by $\chi$ must contain $\psi$ as a constituent, by the Frobenius Reciprocity Theorem. Thus the left side of (3.5) $\sum_{\sigma \in G} \tilde{\chi}(\sigma) \geq \sum_{\sigma \in G} \psi(\sigma) > 0$. So there exists $\pi \in S_m$ such that $\sum_{\sigma \in H_{\alpha\pi}} \chi(\sigma) > 0$. Notice that $m(\alpha\pi) = m(\alpha)$ and hence $(k_1, \ldots, k_n) \in m(\Omega(\chi, H))$.

The last assertion follows readily from the result of Merris stated in (b). $\square$

**Remark 3.2** From the proof one readily sees that if $\alpha \in \hat{\Delta}$ with multiplicity vector $m(\alpha) = (k_1, \ldots, k_n)$, then for any given $\sigma \in S_n$, there is a $\beta \in \hat{\Delta}$ such that $m(\beta) = (m_1(\beta), \ldots, m_n(\beta)) = (k_{\sigma(1)}, \ldots, k_{\sigma(n)})$. This is consistent with Remark 2.6.

The next lemma is due to Horn and Weyl.

**Lemma 3.3** [2, 13] *Suppose $\lambda_1, \ldots, \lambda_n$ are complex numbers with $|\lambda_1| \geq \cdots \geq |\lambda_n|$, and $s_1 \geq \cdots \geq s_n$ are nonnegative real numbers. Then there exists $T \in \mathrm{End}\,(V)$ with singular values $s_1, \ldots, s_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ if and only if $\prod_{j=1}^n |\lambda_j| = \prod_{j=1}^n s_j$ and*

$$\prod_{j=1}^k |\lambda_j| \leq \prod_{j=1}^k s_j \quad \text{for } k = 1, \ldots, n-1.$$

The following characterizations of normal operators are known; for example, see [5].

**Lemma 3.4** *Let $T \in \mathrm{End}\,(V)$. The following are equivalent.*
(a) *$T$ is normal.*
(b) *The moduli of the eigenvalues of $T$ are the singular values of $T$.*
(c) *The sum of the moduli of the eigenvalues of $T$ equals the sum of its singular values.*

**Lemma 3.5** *Suppose $R, S \in \mathrm{End}\,(V)$ have nonnegative eigenvalues $a_1 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq \cdots \geq b_n \geq 0$, respectively, such that $\prod_{j=1}^k a_j \leq \prod_{j=1}^k b_j$ for all $k = 1, \ldots, n$. Then*

$$\mathrm{Tr}\, K(R) \leq \mathrm{Tr}\, K(S).$$

**Proof:** Suppose $R$ and $S$ satisfy the assumption. By Lemma 3.3, there exists a linear operator $T$ on $V$ having singular values $b_1 \geq \cdots \geq b_{n-1} \geq \tilde{b}_n$ and eigenvalues $a_1 \geq \cdots \geq a_n$, where

$$\tilde{b}_n = \begin{cases} (a_1 \ldots a_n)/(b_1 \cdots b_{n-1}) & \text{if } b_n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\begin{aligned} \mathrm{Tr}\, K(R) &= \mathrm{Tr}\, K(T) \qquad \text{(by the construction of } T) \\ &\leq \mathrm{Tr}\, \{K(T)^* K(T)\}^{1/2} \qquad \text{(by Lemma 3.3)} \\ &\leq \mathrm{Tr}\, K(S) \qquad \text{(by the construction of } T). \end{aligned}$$

□

Recall that $m_j(\alpha)$ is the number of occurrence of $j$ in $\alpha \in \hat{\Delta}$.

**Lemma 3.6** *Suppose $\Gamma_{m,r} \cap \bar{\Delta}$ contains an element $\alpha$ with $m_p(\alpha) > m_q(\alpha)$ for some $1 \leq p \neq q \leq r$. Let $R, S \in \mathrm{End}\,(V)$ have eigenvalues $a_1 \geq \cdots \geq a_n > 0$ and $b_1 \geq \cdots \geq b_n > 0$, respectively, with $a_r > 0$, $(b_1, \ldots, b_n)$ is obtained from $(a_1, \ldots, a_n)$ by replacing $(a_j, a_{j+1})$ with $(a_j t, a_{j+1}/t)$ for some $t > 1$ and $1 \leq j < r$. Then $\mathrm{Tr}\, K(R) < \mathrm{Tr}\, K(S)$.*

**Proof:** By Remark 2.6,

$$\operatorname{Tr} K(\operatorname{diag}(x_1, \ldots, x_n)) = \sum_{\alpha \in \hat{\Delta}} \prod_{j=1}^{n} x_j^{m_j(\alpha)}$$

is a symmetric polynomial in $x_1, \ldots, x_n$. Thus $\operatorname{Tr} K(S) - \operatorname{Tr} K(R)$ is a nonnegative combination of terms of the form

$$(a_j t)^{m_j(\alpha)} (a_{j+1}/t)^{m_{j+1}(\alpha)} - a_j^{m_j(\alpha)} a_{j+1}^{m_{j+1}(\alpha)}.$$

By Remark 2.6, there exists $\beta \in \hat{\Delta}$ with $m(\beta) := (m_1(\beta), \ldots, m_n(\beta))$ if and only if there exists $\tilde{\beta} \in \hat{\Delta}$ so that $m(\tilde{\beta})$ is obtained from $m(\beta)$ by switching the $j$th and $(j+1)$st entries ($\beta$ and $\tilde{\beta}$ may be identical and the following $g_k(t)$ is simply zero). So $\operatorname{Tr} K(S) - \operatorname{Tr} K(R)$ is actually a nonnegative combination of terms of the form

$$g_k(t) := (a_h t)^k + (a_{h+1}/t)^k - (a_h^k + a_{h+1}^k) = [a_h^k - (a_{h+1}/t)^k](t^k - 1) \geq 0 \qquad (3.6)$$

with $0 \leq k \leq r$ ($k = |m_h(\alpha) - m_{h+1}(\alpha)|$). Notice that $g_k(t)$ is positive if $k > 0$ and $a_h > 0$. Since $\Gamma_{m,r} \cap \hat{\Delta}$ contains an element $\alpha$ with $m_p(\alpha) > m_q(\alpha)$ for some $1 \leq p \neq q \leq r$, by Remark 2.6, there exists $\beta \in \Gamma_{m,r} \cap \hat{\Delta}$ such that $m_j(\beta) > m_{j+1}(\beta)$. By Remark 2.6 again, there exists $\tilde{\beta} \in \Gamma_{m,r} \cap \hat{\Delta}$ such that $m(\tilde{\beta})$ can be obtained from $m(\beta)$ by switching the $j$th and $j+1$st entries. Clearly $\beta, \tilde{\beta} \in \hat{\Delta}$ are not identical. Set $k_0 := m_j(\beta) - m_{j+1}(\beta)$ and

$$\eta := (a_j a_{j+1})^{m_{j+1}(\beta)} \prod_{i=1, i \neq j, j+1}^{r} a_i^{m_i(\beta)} > 0,$$

since $a_r > 0$. Then

$$\begin{aligned}
0 &< \eta[(a_j t)^{k_0} + (a_{j+1}/t)^{k_0} - (a_j^{k_0} + a_{j+1}^{k_0})] \\
&= [\prod_{i=1}^{r} b_i^{m_i(\beta)} + \prod_{i=1}^{r} b_i^{m_i(\tilde{\beta})}] - [\prod_{i=1}^{r} a_i^{m_i(\beta)} + \prod_{i=1}^{r} a_i^{m_i(\tilde{\beta})}] \\
&\leq \sum_{\alpha \in \hat{\Delta}} \left( \prod_{i=1}^{n} b_i^{m_i(\alpha)} - \prod_{i=1}^{n} a_i^{m_i(\alpha)} \right) \\
&= \operatorname{Tr} K(S) - \operatorname{Tr} K(R).
\end{aligned}$$

Hence, we have $\operatorname{Tr} K(S) > \operatorname{Tr} K(R)$ as asserted. $\square$

# 4 Different Types of Characters

In [7], the authors identified different types of linear characters $\chi$ so that (I) – (III) hold. Here, we extend the results to nonlinear characters. It turns out that the results are similar to the linear case even though the proofs are more involved.

**Theorem 4.1** *Let $\tilde{r}$ be an integer satisfying $\tilde{r} \geq \mu(\bar{\Delta}) > 1$. The following conditions are equivalent.*
(a) *Every $\alpha \in \Gamma_{m,\tilde{r}} \cap \bar{\Delta}$ satisfies $m_1(\alpha) = \cdots = m_{\tilde{r}}(\alpha)$ and $\tilde{r} = \mu(\bar{\Delta})$ (hence $\tilde{r}|m$).*
(b) *There exists a nonnormal $T \in \mathrm{End}\,(V)$ with $\mathrm{rank}\,(T) = \tilde{r}$ such that $K(T)$ is normal.*
(c) *For any $T \in \mathrm{End}\,(V)$ of the form $T_1 \oplus 0$, where $T_1$ is an invertible operator acting on a $\tilde{r}$-dimensional subspace $V_1$ of $V$, the induced operator $K(T)$ is a multiple of an orthogonal projection $P_T$ on $V_\chi^m(H)$.*
*In addition, if (c) holds, then the orthogonal projection $P_T$ is indeed the natural projection from $V_\chi^m(H)$ onto the orthogonal sum $\oplus_{\alpha \in \bar{\Delta} \cap \Gamma_{m,r}} O_\alpha$ with respect to an orthonormal triangularization basis $\{e_1, \ldots, e_r\}$ for $T_1$ in $V_1$, where $r := \mu(\bar{\Delta})$.*

**Proof:** The implication (c) $\Longrightarrow$ (b) is clear.

(b) $\Longrightarrow$ (a): Suppose $T \in \mathrm{End}\,(V)$ is not normal and $\mathrm{rank}\,(T) = \tilde{r}$ so that $K(T)$ is a (nonzero) normal operator. Let $T$ have singular values $s_1 \geq \cdots \geq s_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| \geq \cdots \geq |\lambda_n|$. As $T$ is not normal and has rank $\tilde{r}$, by Lemma 3.4 there is a smallest integer $p$ with $p \leq \tilde{r}$ such that $s_p > |\lambda_p|$. By Lemma 3.3, $\prod_{j=1}^k |\lambda_j| \leq \prod_{j=1}^k s_j$ for $k = 1, \ldots, \tilde{r}$, and $|\prod_{j=1}^n \lambda_j| = \prod_{j=1}^n s_j$. Let $\tilde{D} \in \mathrm{End}\,(V)$ have matrix representation $[\tilde{D}]_{\mathcal{B}} = \mathrm{diag}\,(|\lambda_1|, \ldots, |\lambda_n|)$ with respect to an orthonormal basis $\mathcal{B}$; construct $D \in \mathrm{End}\,(V)$ with $[D]_{\mathcal{B}} = \mathrm{diag}\,(d_1, \ldots, d_n)$ as follows.

1. If $s_p = \cdots = s_{\tilde{r}}$, set $d_{\tilde{r}} = |\lambda_p|$ and $d_k = s_k$ for other $k$. (Note that this can only happen when $\tilde{r} < n$.)

2. If $s_p = \cdots = s_h > s_{h+1}$ for some $h < \tilde{r}$, set $t =: \min\{s_h/|\lambda_h|, \sqrt{s_h/s_{h+1}}\} > 1$, when $\lambda_h \neq 0$, otherwise $t := \sqrt{s_h/s_{h+1}}$, $(d_h, d_{h+1}) = (s_h/t, ts_{h+1})$ and $d_k = s_k$ for other $k$.

In both cases, we have $d_1 \geq \cdots \geq d_n$ and

$$\prod_{j=1}^{k} |\lambda_j| \leq \prod_{j=1}^{k} d_j, \qquad k = 1, \ldots, n-1,$$

and $\prod_{j=1}^n |\lambda_j| = \prod_{j=1}^n d_j$ which is equal to 0 if $\tilde{r} < n$. By Lemma 3.5, we have

$$\mathrm{Tr}\, K(\tilde{D}) \leq \mathrm{Tr}\, K(D).$$

Suppose that (a) were not true. By the definition of $D$ and Lemma 3.6, we have

$$\operatorname{Tr} K(D) < \operatorname{Tr} K(|T|),$$

where $|T|$ is the positive semidefinite square root of $T^*T$, that is, $|T|^2 = T^*T$, and has eigenvalues $s_1, \ldots, s_{\tilde{r}}, 0, \ldots, 0$. As a result,

$$\operatorname{Tr} K(\tilde{D}) < \operatorname{Tr} K(|T|).$$

Since the eigenvalues of $K(\tilde{D})$ are just the moduli of those of $K(T)$, by Lemma 3.4, $K(T)$ is not normal, which is a contradiction.

Finally, since $\Gamma_{m,r} \cap \bar{\Delta} \subseteq \Gamma_{m,\tilde{r}} \cap \bar{\Delta}$, if $\tilde{r} > r := \mu(\bar{\Delta})$ then every element $\alpha \in \Gamma_{m,r} \cap \bar{\Delta} \subseteq \Gamma_{m,\tilde{r}} \cap \bar{\Delta}$ will satisfy $m_1(\alpha) = \cdots = m_r(\alpha) = m_{\tilde{r}}(\alpha) = 0$, which is a contradiction. Thus $\tilde{r} = r$.

(a) $\Longrightarrow$ (c) Suppose (a) holds, and suppose $T = T_1 \oplus 0$, where $T_1$ is an invertible operator acting on a $\tilde{r}$-dimensional subspace $V_1$ of $V$. Then the number of nonzero eigenvalues of $K(T)$ is the same as number of the nonzero singular values of $K(T)$; all the nonzero eigenvalues equal $\det(T_1)^{m/\tilde{r}}$ and all the nonzero singular values equal $|\det(T_1)|^{m/\tilde{r}}$. Thus $K(T)$ is a multiple of an orthogonal projection, that is, (c) holds.

Let $V = V_1 \oplus V_2$ ($\dim V_1 = r$) be an orthogonal sum such that $V_1$ and $V_2$ are invariant under $T$ and the restrictions of $T$ on $V_i$, $i = 1, 2$ are $T_1$ and $0$, respectively. By Schur's triangularization theorem we may let $\{e_1, \ldots, e_r\}$ be an orthonormal basis for $V_1$ such that

$$T_1 e_i = \lambda_i e_i + u_i, \qquad i = 1, \ldots, r,$$

where $u_i \in \operatorname{span}\{e_1, \ldots, e_{i-1}\}$, $i = 2, \ldots, r$ and $u_1 := 0$. Let $\{e_{r+1}, \ldots, e_n\}$ be an orthonormal basis for $V_2$. If $k$ is in the image of $\alpha \in \hat{\Delta}$, where $k = r+1, \ldots, n$, then $K(T)e^*_\alpha = Te_{\alpha(1)} * \cdots * Te_{\alpha(m)} = 0$. When $\alpha \in \hat{\Delta} \cap \Gamma_{m,r}$,

$$\begin{aligned} K(T)e^*_\alpha &= T_1 e_{\alpha(1)} * \cdots * T_1 e_{\alpha(m)} \\ &= (\lambda_{\alpha(1)} e_{\alpha(1)} + u_{\alpha(1)}) * \cdots * (\lambda_{\alpha(1)} e_{\alpha(m)} + u_{\alpha(m)}) \\ &= (\prod_{j=1}^{m} \lambda_{\alpha(j)}) e^*_\alpha + \sum c_\gamma e^*_\gamma, \end{aligned}$$

where the sum is over those $\gamma \in \hat{\Delta} \cap \Gamma_{m,r}$ such that $\gamma(i) \le \alpha(i)$, $1 \le i \le m$ with at least one strict inequality. Since all $\omega \in \hat{\Delta} \cap \Gamma_{m,r}$ satisfy $m(\omega) = (m/r, \ldots, m/r)$, each $e^*_\gamma$ has exactly $m/r$ copies of $e_r$ as constituents which must come from $Te_r$'s in the above expression. Continuing the argument, we conclude that the $m/r$ copies $e_i$ of $e^*_\gamma$ are from $Te_i$, $i = 1, \ldots, r$. This implies $e^*_\gamma = e^*_\alpha$, a contradiction. So $K(T)e^*_\alpha = (\det T_1)^{m/r} e^*_\alpha$,

$\alpha \in \hat{\Delta} \cap \Gamma_{m,r}$. Thus $K(T)$ is $(\det T_1)^{m/r} I$ while restricted to $\sum_{\alpha \in \bar{\Delta} \cap \Gamma_{m,r}} O_\alpha$ and 0 while restricted to the orthogonal complement of $\sum_{\alpha \in \bar{\Delta} \cap \Gamma_{m,r}} O_\alpha$. □

Applying Theorem 4.1 with $\tilde{r} = n$, we get the following corollary (cf. Theorem 1.1 in [9, Chapter 6]).

**Corollary 4.2** *Let $H$ and $\chi$ be given. The following conditions are equivalent.*
(a) *Every element $\alpha \in \bar{\Delta}$ satisfies $m_1(\alpha) = \cdots = m_n(\alpha) = m/n$.*
(b) *$K(T) = \xi_T I$, for any $T \in \mathrm{Aut}(V)$.*
*In addition, if the (b) holds, then $\xi_T = (\det T)^{m/n} I$.*

**Proof:** The orthogonal projection from $V_\chi^m(H)$ into itself is a scalar multiple of the identity map. So by Theorem 4.1, we conclude the equivalence of (a) and (b). Now let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $T$. If (a) holds, the eigenvalues of $K(T)$ are equal to $(\det T)^{m/n}$. Thus $\xi_T = (\det T)^{m/n}$.

**Definition 4.3** In the following, we say that $(\chi, n)$ is of *determinant type* if any one (and hence all) of the conditions (a) – (c) in Corollary 4.2 holds, with $\mu(\bar{\Delta}) > 1$. Furthermore, we say that $(\chi, n)$ is of the *special type* if any one (and hence all) the conditions (a) – (c) in Theorem 4.1 holds with $\mu(\bar{\Delta}) > 1$; otherwise, we say that $(\chi, n)$ is of the *general type*. Notice that the determinant type is a particular case of special type.

Note that the alternate character on $S_n$ with $\dim V = n > 1$ is of the determinant type ($\bar{\Delta} = \{(1, \ldots, n)\}$ and $\mu(\bar{\Delta}) = n$); the alternate character on $S_m$ with $1 < m < n$ and $\dim V = n$, is of the special type but not of the determinant type ($\bar{\Delta} = Q_{m,n}$ and $\mu(\bar{\Delta}) = m$); and the principal character is of the general type ($\mu(\bar{\Delta}) = 1$ since $G_{m,n} \subset \bar{\Delta}$ [8, p.108] for all $n \geq 1$). Here we give some additional examples of $(\chi, n)$ that are of the special type, determinant type, and the general type.

**Example 4.4** Consider the alternating group $A_4$ in $S_4$ and use the linear character $\chi_2$ in [3, p.181], that is, $\chi_2(\sigma) = 1$ if $\sigma$ is the identity or a product of two disjoint transpositions, and if $1 \leq i < j < k \leq 4$ then $\chi_2((i, j, k)) = \omega$ and $\chi_2((i, k, j)) = \omega^2$, where $\omega = e^{2\pi i/3}$.

(a) If $n = 2$, then

$$\hat{\Delta} = \bar{\Delta} = \{(1, 1, 2, 2)\},$$

and $(\chi_2, 2)$ is of the determinant type since $\mu(\bar{\Delta}) = 2 = n$, $\Gamma_{4,2} \cap \bar{\Delta} = \{(1, 1, 2, 2)\}$.

(b) If $n = 3$, then

$$\hat{\Delta} = \bar{\Delta} = \{(j, j, k, k) : 1 \leq j < k \leq 3\} \cup \{(1, 1, 2, 3), (1, 2, 2, 3), (1, 2, 3, 3)\}.$$

Now $(\chi_2, 3)$ is of the special type since $\mu(\bar{\Delta}) = 2$, $\Gamma_{4,2} \cap \bar{\Delta} = \{(j, j, k, k) : 1 \leq j < k \leq 3\}$ but not of the determinant type.

(c) In general, when $n \geq 4 = m$, then

$$\{(j, j, k, k) : 1 \leq j < k \leq n\} \subset \bar{\Delta},$$

and $\bar{\Delta}$ does not contain other $\alpha$ whose image has order 2. So $(\chi_2, n)$ is of the special type since $\mu(\bar{\Delta}) = 2$, $\Gamma_{4,2} \cap \bar{\Delta} = \{(j, j, k, k) : 1 \leq j < k \leq n\}$ but not of the determinant type. So besides the exterior spaces $\wedge^m V$ ($\dim V = n$), we have other special types $(\chi, n)$ with $m < n$.

**Remark 4.5** If we require $n = m$, then $(\epsilon, n)$ is the only determinant type, where $\epsilon : S_n \to \mathbb{C}$ is the alternate character. It is because for any irreducible character $\chi$ on $H$ with $m = n$, $\{(1, 2, \ldots, n)\} = Q_{n,n} \subset \bar{\Delta}$. If $(\chi, n)$ is of determinant type, then $\bar{\Delta} = Q_{n,n}$ otherwise there is $(i_1, i_2, \ldots, i_n)(\neq (1, 2, \ldots, n)) \in \bar{\Delta}$. By Corollary 4.2, $(i_1, \ldots, i_n)$ is a rearrangement of $(1, 2, \ldots, n)$. Let $\{e_1, \ldots, e_n\}$ be a basis for $V$ and let $T \in \mathrm{End}\,(V)$ such that $Te_k = e_{i_k}$, $k = 1, \ldots, n,$. Then $K(T)e_1 * \cdots * e_n = e_{i_1} * \cdots * e_{i_n}$ and the vectors $e_1 * \cdots * e_n$ and $e_{i_1} * \cdots * e_{i_n}$ are linearly independent by (2.2). Thus $K(T)$ is not a scalar multiple of the identity, contradicting Corollary 4.2 (b). Now $\bar{\Delta} = Q_{n,n}$ and hence $H = S_n$ and $\chi = \epsilon$ [12, p.96].

**Example 4.6** Consider $S_3$ and use the (only) nonlinear irreducible character $\chi_3$ in [3, p.157], that is, $\chi_3(e) = 2$, $\chi_3((12)) = 0$, $\chi_3((123)) = -1$.

(a) If $n = 2$, then

$$\bar{\Delta} = \{(1, 1, 2), (1, 2, 2)\}, \quad \hat{\Delta} = \{(1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 2)\},$$

and $(\chi_3, 2)$ is of the general type.

(b) [12] If $n = 3$, then

$$\bar{\Delta} = \{(1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3), (1, 3, 3), (2, 2, 3), (2, 3, 3)\},$$

and

$$\begin{aligned} \hat{\Delta} \;=\; & \{(1, 1, 2), (1, 2, 1); (1, 1, 3), (1, 3, 1); (1, 2, 2), (2, 1, 2); (1, 2, 3), (1, 3, 2), \\ & (2, 1, 3), (2, 3, 1); (1, 3, 3), (3, 1, 3); (2, 2, 3), (2, 3, 2); (2, 3, 3), (3, 2, 3)\}, \end{aligned}$$

and $(\chi_3, 3)$ is of the general type.

## 5 Operator properties

In the following, we characterize $T \in \mathrm{End}\,(V)$ so that $K(T)$ is normal. We exclude the trivial case when $K(T) = 0$, or equivalently, when $T \in \mathrm{End}\,(V)$ has $\mathrm{rank}\,(T) < \mu(\bar{\Delta})$.

**Theorem 5.1** *Let $r = \mu(\bar{\Delta})$ and $T \in \mathrm{End}\,(V)$ with $\mathrm{rank}\,(T) \geq r$. Then $K(T)$ is normal if and only if one of the following holds.*
(a) *$T$ is normal.*
(b) *$(\chi, n)$ is of the special type, and $T = T_1 \oplus 0$, where $T_1$ is an invertible nonnormal operator acting on an $r$-dimensional subspace $V_1$ of $V$.*

**Proof:** If (a) holds, then $K(T)$ is normal by Proposition 2.5 (f). If (b) holds, then $K(T)$ is normal by Theorem 4.1 (c).

Conversely, let $T \in \mathrm{End}\,(V)$ satisfy $\tilde{r} := \mathrm{rank}\,(T) \geq r$ so that $K(T)$ is a nonzero normal operator. Assume that (a) does not hold, that is, (b) of Theorem 4.1 holds. Thus by Theorem 4.1 $\mathrm{rank}\,(T) = r$ and $T$ has the desired form. □

**Corollary 5.2** *Suppose $(\chi, n)$ is not of the determinant type, and $T \in \mathrm{End}\,(V)$ is invertible. Then $T$ is normal if and only if $K(T)$ is normal.*

**Proof:** The necessity part is clear. To prove the converse, suppose $K(T)$ is normal. Then Theorem 5.1 (a) or (b) holds. Since $\mathrm{rank}\,(T) = n$, if (b) holds, then $(\chi, n)$ is of the determinant type by Theorem 4.1, contradicting the assumption on $(\chi, n)$. Hence, we see that (a) holds, and the result follows. □

By Corollary 4.2, if every element $\alpha$ in $\bar{\Delta}$ satisfies $m_1(\alpha) = \cdots = m_n(\alpha)$, then $n = \mu(\bar{\Delta})$ and $K(T) = \det\,(T)^{m/n} I$ for all $T \in \mathrm{End}\,(V)$. Consequently, $K(T)$ is positive definite (respectively, unitary) if and only if $\det\,(T)^{m/n} > 0$ (respectively, $|\det\,(T)| = 1$). Apart from this trivial case, we will show that $K(T)$ is a nonzero multiple of a positive definite (respectively, unitary) operator if and only if $T$ is.

**Theorem 5.3** *Suppose $(\chi, n)$ is not of the determinant type, and $T \in \mathrm{End}\,(V)$. If there exists $\eta \in \mathbb{C}$ with $|\eta| = 1$ such that $\eta K(T)$ is positive definite then there exists $\xi \in \mathbb{C}$ with $\xi^m = \eta$ such that $\xi T$ is positive definite.*

**Proof:** The sufficiency part is clear. To prove the converse, suppose $\eta K(T)$ is positive definite, where $\eta \in \mathbb{C}$ with $|\eta| = 1$. Then $K(T)$ is normal. Since $(\chi, n)$ is not of the determinant type and $K(T)$ is invertible, by Corollary 5.2 we see that $T$ is normal. Furthermore, if $T$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then none of them is zero by Remark 2.6.

To complete the proof, we show that there exists $\xi \in \mathbb{C}$ such that $\xi^m = \eta$ and $\lambda_j = \xi|\lambda_j|$ for all $1 \leq j \leq n$, as follows. Since $(\chi, n)$ is not of the determinant type, there exists $\alpha \in \bar{\Delta}$

such that $m_p(\alpha) < m_q(\alpha)$ for some $1 \le p < q \le n$. By Lemma 3.1, there exists $\beta \in \bar{\Delta}$ such that $(m_p(\beta), m_q(\beta)) = (m_p(\alpha)+1, m_q(\alpha)-1)$ and $m_t(\beta) = m_t(\alpha)$ for $t \ne p, q$. Now, for any $1 < j \le n$, there exists a permutation $\sigma \in S_n$ so that $\sigma(p) = 1$ and $\sigma(q) = j$. Furthermore, by Proposition 2.5 (g), we see that $\lambda_\alpha = \prod_{i=1}^n \lambda_{\sigma(i)}^{m_i(\alpha)}$ and $\lambda_\beta = \prod_{i=1}^n \lambda_{\sigma(i)}^{m_i(\beta)}$ are eigenvalues of $K(T)$. It follows that $\eta\lambda_\alpha$ and $\eta\lambda_\beta$ are eigenvalues of $\eta K(T)$, and hence both of them are positive. Consequently,

$$\lambda_\alpha/\lambda_\beta = \lambda_j/\lambda_1 > 0, \qquad 1 \le j \le n.$$

Thus, all the eigenvalues of $T$ have the same argument, that is, $\xi T$ is positive definite for some $\xi \in \mathbb{C}$ with $|\xi| = 1$. Since both $K(\xi T) = \xi^m K(T)$ and $\eta K(T)$ are positive definite, we see that $\xi^m = \eta$ as asserted. $\square$

**Theorem 5.4** *Suppose $(\chi, n)$ is not of the determinant type. Let $T \in \mathrm{End}\,(V)$. Then $K(T)$ is unitary or a nonzero scalar if and only if $T$ has the corresponding property.*

**Proof:** The sufficiency part is clear. To prove the converse, suppose $K(T)$ is unitary (respectively, a scalar). Since $(\chi, n)$ is not of the determinant type and $K(T)$ is normal, by Corollary 5.2 we see that $T$ is normal. Suppose $T$ has eigenvalues $\lambda_1, \dots, \lambda_n$. One can use arguments similar to those in the proof of Theorem 5.3 to show that $|\lambda_j/\lambda_1| = 1$ (respectively, $\lambda_j/\lambda_1 = 1$) for all $j = 2, \dots, n$. The result follows. $\square$

**Theorem 5.5** *Let $r = \mu(\bar{\Delta})$ and $T \in \mathrm{End}\,(V)$ with $\mathrm{rank}\,(T) \ge r$. If there exists $\eta \in \mathbb{C}$ with $|\eta| = 1$ such that $\eta K(T)$ is* (i) *Hermitian,* (ii) *positive semi-definite, or* (iii) *an orthogonal projection, then one of the following holds.*

(a) *There exists $\xi \in \mathbb{C}$ with $\xi^m = \eta$ such that $\xi T$ has the corresponding property.*

(b) *$(\chi, n)$ is of the special type, and $T = T_1 \oplus 0$, where $T_1$ is an invertible operator acting on an $r$-dimensional subspace $V_1$ of $V$, and $\eta \det\,(T_1)^{m/r}$ is* (i) *real,* (ii) *positive, or* (iii) *equal to 1.*

**Proof:** The sufficiency part is clear. Conversely, if $K(T)$ satisfies (i), (ii), or (iii), then $K(T)$ is normal. Hence $T$ satisfies condition (a) or (b) of Theorem 5.1. If Theorem 5.1 (a) holds, then $T$ is normal. One can use arguments similar to those in the proof of Theorem 5.3 to show that condition (a) of this theorem holds. If Theorem 5.1 (b) holds, one easily checks that condition (b) of Theorem 5.5 holds.

$\square$

Note that Theorem 5.5 (i) covers the special cases when $K(T)$ is Hermitian or skew-Hermitian.

## 6 Equality of induced operators

We now determine the conditions for two induced operators to be equal.

**Theorem 6.1** *Let $r = \mu(\bar{\Delta})$. Then $S, T \in \mathrm{End}\,(V)$ satisfy $K(S) = K(T)$ if and only if one of the following holds.*
(a) $\mathrm{rank}\,(S) < r$ *and* $\mathrm{rank}\,(T) < r$.
(b) *There exists $\xi \in \mathbb{C}$ with $\xi^m = 1$ such that $S = \xi T$.*
(c) *$(\chi, n)$ is of the special type, and there are unitary operators $U, V \in \mathrm{End}\,(V)$ such that $USV = S_1 \oplus 0$ and $UTV = T_1 \oplus 0$, where $S_1$ and $T_1$ acting on an $r$-dimensional subspace $V_1$ of $V$, and $\det\,(S_1)^{m/r} = \det\,(T_1)^{m/r}$.*

**Proof:** If (a) or (b) holds, then clearly $K(S) = K(T)$. If (c) holds, then by Theorem 4.1

$$K(U)K(S)K(V) = K(S_1 \oplus 0) = K(T_1 \oplus 0) = K(U)K(T)K(V)$$

and hence $K(S) = K(T)$. Conversely, suppose $K(S) = K(T)$. If $K(S) = K(T) = 0$, then (a) holds. Otherwise, let $U, V \in \mathrm{End}\,(V)$ be unitary such that $USV$ has matrix representation $[USV]_{\mathcal{B}} = \mathrm{diag}\,(a_1, \ldots, a_k, 0, \ldots, 0)$ with respect to an orthonormal basis $\mathcal{B} = \{e_1, \ldots, e_n\}$, where $k \geq r$ and $a_1 \geq \cdots \geq a_k > 0$. Suppose $D, P \in \mathrm{End}\,(V)$ are such that

$$[D]_{\mathcal{B}} = \mathrm{diag}\,(1/a_1, \ldots, 1/a_k) \oplus I_{n-k} \qquad \text{and} \qquad [P]_{\mathcal{B}} = I_k \oplus 0_{n-k}.$$

Then

$$\begin{aligned} K(P) &= K(USVD) \\ &= K(U)K(S)K(V)K(D) \\ &= K(U)K(T)K(V)K(D) \\ &= K(UTVD) \end{aligned}$$

is an orthogonal projection. Then Theorem 5.5 (a) or (b.iii) holds.

Case 1. If Theorem 5.5 (b.iii) holds for $UTVD$, that is, $k = r$ and $[UTVD]_{\mathcal{B}}$ is unitarily similar to $C \oplus 0_{n-r} \in M_n$ such that $\det\,(C)^{m/r} = 1$. Suppose

$$[UTVD]_{\mathcal{B}} = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

such that $C_1 \in M_r$. Now

$$K(P) = K(P^3) = (K(P))^3 = K(P)K(UTVD)K(P) = K(PUTVDP) = K\left(\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix}\right).$$

Using the fact that $(\chi, n)$ is of the special type, we see that

$$1 = (\zeta_1 \cdots \zeta_r)^{m/r} = \det(C_1)^{m/r},$$

where $\zeta_1, \ldots, \zeta_r$ are the eigenvalues of $C_1$. Thus $|\det(C_1)| = |\det(C)|$ is the product of the $r$ largest singular values of $C \oplus 0_{n-r}$, which is unitarily similar to $[UTVD]_{\mathcal{B}}$. By [4, Theorem 4], we see that $[UTVD]_{\mathcal{B}} = C_1 \oplus C_4$. Since $UTVD$ has rank $r$, we see that $C_4 = 0_{n-r}$. Thus condition (c) of the theorem holds with $[USV]_{\mathcal{B}} = A_1 \oplus 0_{n-r}$ and $[UTV]_{\mathcal{B}} = [UTVD]_{\mathcal{B}}[D]_{\mathcal{B}}^{-1} = C_1 A_1 \oplus 0_{n-r}$, where $A_1 = \operatorname{diag}(a_1, \ldots, a_r)$.

Case 2. If Theorem 5.5 (a) holds for $UTVD$, that is, $\zeta UTVD$ is an orthogonal projection for some $\zeta \in \mathbb{C}$ with $\zeta^m = 1$. Suppose

$$[\zeta UTVD]_{\mathcal{B}} = \begin{pmatrix} C_1 & C_2 \\ C_2^* & C_3 \end{pmatrix}$$

with $C_1 \in M_k$. We claim that $C_1 = I_k$, and hence $[\zeta UTVD]_{\mathcal{B}} = I_k \oplus 0_{n-k}$. If our claim were not true, then $C_1$ has eigenvalues $c_1 \geq \cdots \geq c_k$ with $1 \geq c_1$ and $1 > c_k \geq 0$. Since $[P(\zeta UTVD)P]_{\mathcal{B}} = C_1 \oplus 0_{n-k}$, it follows from Theorem 5.5 that $K(P(\zeta UTVD)P)$ is not an orthogonal projection. But then

$$K(P) = K(P^3) = (K(P))^3 = K(P)K(\zeta UTVD)K(P) = K(P(\zeta UTVD)P),$$

which is a contradiction. Thus our claim is proved and hence $[USV]_{\mathcal{B}} = A_1$ and $[\zeta UTV]_{\mathcal{B}} = [\zeta UTVD]_{\mathcal{B}}[D]_{\mathcal{B}}^{-1} = A_1$, where $A_1 = \operatorname{diag}(a_1, \ldots, a_k, 0, \ldots, 0)$, that is, condition (b) of the theorem holds with $\xi = \zeta$. $\square$

The results in this sections explain why if $\chi$ is the principal character or if $\operatorname{rank}(T) > m$, then (I) – (III) hold. Also, one sees why (I) – (III) fail if $\chi$ is the alternate character on $S_m$. In particular, we have the following corollary.

**Corollary 6.2** *Suppose $(\chi, n)$ is not of the determinant type.*
(a) *Let $S, T \in \operatorname{End}(V)$ be such that*

$$\operatorname{rank}(T) \geq \begin{cases} \mu(\bar{\Delta}) + 1 & \textit{if } (\chi, n) \textit{ is of the special type,} \\ \mu(\bar{\Delta}) & \textit{otherwise.} \end{cases}$$

*Then* (I) – (III) *hold.*
(b) *If $(\chi, n)$ is of the special type, then there exist $S, T \in \operatorname{End}(V)$ (with ranks equal to $\mu(\bar{\Delta})$) such that all of* (I) – (III) *fail.*

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