Math 2660 Topics in Linear Algebra, Key

6.3

1a,b,d,2a,b,d,3a,b,5,6

1 (a) $\det(A-\lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - 1 = (\lambda-1)(\lambda+1)$. So the eigenvalues of A are $1, -1$.

Case 1: $\lambda = 1$, consider $(A-I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - I|\mathbf{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_1 = x_2 = t$. Thus $\mathbf{x} = t(1,1)^T$ are eigenvectors of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(1,1)^T\}$. Pick $\mathbf{x}_1 = (1,1)^T$.

Case 2: $\lambda = -1$, consider $(A+I)\mathbf{x} = \mathbf{0}$. Solving

$$[A + I|\mathbf{0}] = \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -t$. Thus $\mathbf{x} = t(-1,1)^T$ are eigenvectors of A corresponding to $\lambda = -1$ when $t \neq 0$. Pick $\mathbf{x}_2 = (-1,1)^T$.

Set $X = [\mathbf{x}_1\ \mathbf{x}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $D = \text{diag}\,(1,-1)$. Then $AX = XD$, i.e., $A = XDX^{-1}$.

(b) $\det(A-\lambda I) = \det \begin{bmatrix} 5-\lambda & 6 \\ -2 & -2-\lambda \end{bmatrix} = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$. So the eigenvalues of A are $1, 2$.

Case 1: $\lambda = 1$, consider $(A-I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - I|\mathbf{0}] = \left[\begin{array}{cc|c} 4 & 6 & 0 \\ -2 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -\frac{3}{2}t$. Thus $\mathbf{x} = \frac{t}{2}(-3,2)^T$ are eigenvectors of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(-3,2)^T\}$. Pick $\mathbf{x}_1 = (-3,2)^T$.

Case 2: $\lambda = 2$, consider $(A-2I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{cc|c} 3 & 6 & 0 \\ -2 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -2t$. Thus $\mathbf{x} = t(-2,1)^T$ are eigenvectors of A corresponding to $\lambda = -1$ when $t \neq 0$. Pick $\mathbf{x}_2 = (-2,1)^T$.

Set $X = [\mathbf{x}_1\ \mathbf{x}_2] = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$ and $D = \text{diag}\,(1,2)$. Then $AX = XD$, i.e., $A = XDX^{-1}$.

(d) Since A is triangular, the eigenvalues are the diagonal entries of A., i.e., the eigenvalues of A are $2, 1, -1$

Case 1: $\lambda = 2$, consider $A\mathbf{x} = \mathbf{0}$. Solving

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = 0$, $x_2 = 0$, $x_1 = t$. Thus $x = t(1,0,0)^T$ are eigenvectors of A corresponding to $\lambda = 2$ when $t \neq 0$. So the eigenspace is span $\{(1,0,0)^T\}$. Pick $\mathbf{x}_1 = (1,0,0)^T$.
Case 2: $\lambda = 1$, consider $A\mathbf{x} = \mathbf{0}$. Solving

$$[A - I|\mathbf{0}] = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = 0$, $x_2 = t$, $x_1 = -2t$. Thus $x = t(-2,1,0)^T$ are eigenvectors of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(-2,1,0)^T\}$. Pick $\mathbf{x}_2 = (-2,1,0)^T$.
Case 3: $\lambda = -1$, consider $A\mathbf{x} = \mathbf{0}$. Solving

$$[A + I|\mathbf{0}] = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = t$, $x_2 = -t$, $x_1 = \frac{t}{3}$. Thus $x = \frac{t}{3}(1,-3,3)^T$ are eigenvectors of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(1,-3,3)^T\}$. Pick $\mathbf{x}_3 = (1,-3,3)^T$.

Thus $X = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$ and $D = \operatorname{diag}(2,1,-1)$. Then $AX = XD$, i.e., $A = XDX^{-1}$.

2 (a) Notice that $X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $D = \operatorname{diag}(1,-1)$. So

$$A^6 = (XDX^{-1})^6 = XD^6X^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} I \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = A.$$

Or by direct computation to get $A^6 = I$. Indeed $A^{2n} = I$ and $A^{2n+1} = I$.

(b) Notice that $X = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$, $D = \operatorname{diag}(1,2)$. So

$$A^6 = (XDX^{-1})^6 = XD^6X^{-1} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2^6 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

(d) Notice that $X = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$, $D = \operatorname{diag}(2,1,-1)$. So

$$A^6 = (XDX^{-1})^6 = XD^6X^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2^6 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 3 & 6 & 5 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

3 (a) Notice that $X = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $D = \operatorname{diag}(1,-1)$. So

$$\begin{aligned} A^{-1} &= (XDX^{-1})^{-1} = XD^{-1}X^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^{-1} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = A. \end{aligned}$$

(b) Notice that $X = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$, $D = \text{diag}\,(1, 2)$. So

$$A^{-1} = (XDX^{-1})^{-1} = XD^{-1}X^{-1} = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}.$$

(d) Notice that $X = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & -3 \end{bmatrix}$, $D = \text{diag}\,(2, 1, -1)$. So

$$A^{-1} = (XDX^{-1})^{-1} = XD^{-1}X^{-1} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 3 & 6 & 5 \\ 0 & 3 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

5 $A = XDX^{-1}$ implies $A^T = (XDX^{-1})^T = (X^{-1})^T D X^T$. Set $Y = (X^{-1})^T$. Then $A^T = YDY^{-1}$.

6 Since A is diagonalizable with eigenvalues ± 1, $A = XDX^{-1}$ and $D = \text{diag}\,(\pm 1, \ldots, \pm 1)$ so that $D^{-1} = D$. Thus

$$A^{-1} = (XDX^{-1})^{-1} = XD^{-1}X^{-1} = XDX^{-1} = A.$$