Math 2660 Topics in Linear Algebra, Key

6.2

1a,b,2a

1 (a) The differential equations yield $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$. So $\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{bmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. The eigenvalues of A are $2, 3$.

Case 1: $\lambda = 2$, consider $(A - 2I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -2 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_1 = x_2 = t$. Thus $x = t(1,1)^T$ are eigenvector of A corresponding to $\lambda = 2$ when $t \neq 0$. So the eigenspace is span $\{(1,1)^T\}$. Pick $\mathbf{x}_1 = (1,1)^T$.

Case 2: $\lambda = 3$, consider $(A - 3I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - 3I|\mathbf{0}] = \left[\begin{array}{cc|c} -2 & 1 & 0 \\ -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = \frac{1}{2}t$. Thus $x = 2t(1,2)^T$ are eigenvector of A corresponding to $\lambda = 2$ when $t \neq 0$. So the eigenspace is span $\{(1,2)^T\}$. Pick $\mathbf{x}_2 = (1,2)^T$.

Thus the solution is $\mathbf{y} = c_1 e^{3t}\mathbf{x}_1 + c_2 e^{2t}\mathbf{x}_2 = \begin{bmatrix} c_1 e^{3t} + c_2 e^{2t} \\ c_1 e^{3t} + 2c_2 e^{2t} \end{bmatrix}$, where c_1, c_2 are constants.

(b) The differential equations yield $A = \begin{bmatrix} 2 & 4 \\ -1 & -3 \end{bmatrix}$. So $\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 4 \\ -1 & -3-\lambda \end{bmatrix} = (2-\lambda)(-3-\lambda) + 4 = \lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1)$. The eigenvalues of A are $-2, 1$.

Case 1: $\lambda = -2$, consider $(A + 2I)\mathbf{x} = \mathbf{0}$. Solving

$$[A + 2I|\mathbf{0}] = \left[\begin{array}{cc|c} 4 & 4 & 0 \\ -1 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -t$. Thus $x = t(-1,1)^T$ are eigenvector of A corresponding to $\lambda = -2$ when $t \neq 0$. So the eigenspace is span $\{(-1,1)^T\}$. Pick $\mathbf{x}_1 = (-1,1)^T$.

Case 2: $\lambda = 1$, consider $(A - I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - I|\mathbf{0}] = \left[\begin{array}{cc|c} 1 & 4 & 0 \\ -1 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -4t$. Thus $x = t(-4,1)^T$ are eigenvector of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(-4,1)^T\}$. Pick $\mathbf{x}_2 = (-4,1)^T$.

Thus the solution is $\mathbf{y} = c_1 e^{-2t}\mathbf{x}_1 + c_2 e^{t}\mathbf{x}_2 = \begin{bmatrix} -c_1 e^{-2t} - 4c_2 e^{2t} \\ c_1 e^{-2t} + c_2 e^{t} \end{bmatrix}$, where c_1, c_2 are constants.

2 (a) The differential equations yield $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ with $y_1(0) = 3$, $y_2(0) = 1$.

So $\det(A - \lambda I) = \det \begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = (-1-\lambda)(-1-\lambda) - 4 = \lambda^2 + 2\lambda - 3 =$

$(\lambda - 1)(\lambda + 3)$. The eigenvalues of A are $1, -3$.
Case 1: $\lambda = 1$, consider $(A - I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_1 = x_2 = t$. Thus $x = t(1,1)^T$ are eigenvector of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(1,1)^T\}$. Pick $\mathbf{x}_1 = (1,1)^T$.
Case 2: $\lambda = -3$, consider $(A + 3I)\mathbf{x} = \mathbf{0}$. Solving

$$[A + 3I|\mathbf{0}] = \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -t$. Thus $x = t(-1,1)^T$ are eigenvector of A corresponding to $\lambda = -3$ when $t \neq 0$. So the eigenspace is span $\{(-1,1)^T\}$. Pick $\mathbf{x}_2 = (-1,1)^T$.
Thus the solution is $\mathbf{y} = c_1 e^t \mathbf{x}_1 + c_2 e^{-3t} \mathbf{x}_2 = \begin{bmatrix} c_1 e^t - c_2 e^{-3t} \\ c_1 e^t + c_2 e^{-3t} \end{bmatrix}$, where c_1, c_2 are constants.
Since the initial conditions are given: $y_1(0) = 3$, $y_2(0) = 1$, we have

$$\begin{aligned} c_1 - c_2 &= 3 \\ c_1 + c_2 &= 1 \end{aligned}$$

Solving the system to have $c_1 = 2$, $c_2 = -1$. Thus

$$\mathbf{y} = \begin{bmatrix} 2e^t + e^{-3t} \\ 2e^t - e^{-3t} \end{bmatrix}$$