Math 2660 Topics in Linear Algebra, Key

6.1

1a,b,f,g,2,3,4

1 (a) $\det(A-\lambda I) = \det \begin{bmatrix} 3-\lambda & 2 \\ 4 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda)-8 = \lambda^2-4\lambda-5 = (\lambda-5)(\lambda+1)$.
So the eigenvalues of A are $5, -1$.
Case 1: $\lambda = 5$, consider $(A-5I)\mathbf{x} = \mathbf{0}$. Solving

$$[A-5I|\mathbf{0}] = \left[\begin{array}{cc|c} -2 & 2 & 0 \\ 4 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_1 = x_2 = t$. Thus $\mathbf{x} = t(1,1)^T$ are eigenvectors of A corresponding to $\lambda = 5$ when $t \neq 0$. So the eigenspace is span $\{(1,1)^T\}$.
Case 2: $\lambda = -1$, consider $(A+I)\mathbf{x} = \mathbf{0}$. Solving

$$[A+I|\mathbf{0}] = \left[\begin{array}{cc|c} 4 & 2 & 0 \\ 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = -\frac{1}{2}t$. Thus $\mathbf{x} = 2t(-1,2)^T$ are eigenvectors of A corresponding to $\lambda = -1$ when $t \neq 0$. So the eigenspace is span $\{(-1,2)^T\}$. The book's answer is span $\{(1,-2)^T\}$ and is the same.

(b) $\det(A-\lambda I) = \det \begin{bmatrix} 6-\lambda & -4 \\ 3 & -1-\lambda \end{bmatrix} = (6-\lambda)(-1-\lambda)+12 = \lambda^2-5\lambda+6 = (\lambda-3)(\lambda-2)$.
So the eigenvalues of A are $3, 2$.
Case 1: $\lambda = 3$, consider $(A-3I)\mathbf{x} = \mathbf{0}$. Solving

$$[A-5I|\mathbf{0}] = \left[\begin{array}{cc|c} 3 & -4 & 0 \\ 3 & -4 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$, $x_1 = \frac{3}{4}t$. Thus $x = \frac{1}{4}t(4,3)^T$ are eigenvector of A corresponding to $\lambda = 3$ when $t \neq 0$. So the eigenspace is span $\{(4,3)^T\}$.
Case 2: $\lambda = 2$, consider $(A-2I)\mathbf{x} = \mathbf{0}$. Solving

$$[A+I|\mathbf{0}] = \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 3 & -3 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = x_1 = t$. Thus $\mathbf{x} = t(1,1)^T$ are eigenvectors of A corresponding to $\lambda = 2$ when $t \neq 0$. So the eigenspace is span $\{(1,1)^T\}$.

(f) Since A is triangular, the eigenvalues are the diagonal entries of A., i.e., 0 is the eigenvalue of A of multiplicity 3.
Consider $A\mathbf{x} = \mathbf{0}$. Solving

$$[A|\mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_2 = x_3 = 0$, $x_1 = t$. Thus $\mathbf{x} = t(1,0,0)^T$ are eigenvectors of A corresponding to $\lambda = 0$ when $t \neq 0$. So the eigenspace is span $\{(1,0,0)^T\}$. Though the matrix is 3×3 the dimension of the eigenspace is 1. This reflects the deficiency of the matrix.

(g) Since A is triangular, the eigenvalues are the diagonal entries of A., i.e., the eigenvalues of A are 1 (of multiplicity 2) and 2.

Case 1: $\lambda = 1$, consider $(A - I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - I|\mathbf{0}] = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = t$, $x_2 = -t$, $x_1 = s$. Thus $\mathbf{x} = s(1,0,0)^T + t(0,-1,1)^T$ are eigenvectors of A corresponding to $\lambda = 1$ when $t \neq 0$. So the eigenspace is span $\{(1,0,0)^T, (0,-1,1)^T\}$.

Case 2: $\lambda = 1$, consider $(A - 2I)\mathbf{x} = \mathbf{0}$. Solving

$$[A - 2I|\mathbf{0}] = \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = 0$, $x_2 = t$, $x_1 = t$. Thus $\mathbf{x} = t(1,1,0)^T$ are eigenvectors of A corresponding to $\lambda = 2$ when $t \neq 0$. So the eigenspace is span $\{(1,1,0)^T\}$.

2 If A is triangular, then $A - \lambda I$ is triangular so that $\det(A - \lambda I) = (a_{11} - \lambda) \cdots (a_{nn} - \lambda)$. So the roots to $\det(A - \lambda I) = 0$ are $a_{11}, \ldots, a_{nn}$, i.e., the diagonal entries.

3 A is singular if and only if $\det A = 0$. But $\det A = 0$ means $\det(A - 0 \cdot I) = 0$, i.e., 0 is an eigenvalue of A.

4 If λ is an eigenvalue of a nonsingular A, then $\lambda \neq 0$ by Exercise 3. Let $\mathbf{x}$ be an eigenvector of A corresponding to λ. Then $A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow \mathbf{x} = A^{-1}\lambda\mathbf{x} = \lambda A^{-1}\mathbf{x}$ or $\lambda^{-1}\mathbf{x} = A^{-1}\mathbf{x}$. In other words, $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

6 $A\mathbf{x} = \lambda\mathbf{x} \Rightarrow A^2\mathbf{x} = \lambda A\mathbf{x} = \lambda^2\mathbf{x}$. On the other hand $A^2 = A$ so that $A^2\mathbf{x} = A\mathbf{x} = \lambda\mathbf{x}$. Thus $\lambda\mathbf{x} = \lambda^2\mathbf{x}$. Since $\mathbf{x} \neq 0$, we have $\lambda = \lambda^2$. Thus $\lambda = 0, 1$.