Math 2660 Topics in Linear Algebra, Key

5.6

1,2,3

1 (a) $r_{11} = \|\mathbf{a}_1\| = \|(-1,1)^T\| = \sqrt{2}$
$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \frac{1}{\sqrt{2}}(-1,1)^T$
$r_{12} = \mathbf{q}_1^T\mathbf{a}_2 = \frac{1}{\sqrt{2}}(-1,1)(3,5)^T = \frac{2}{\sqrt{2}} = \sqrt{2}$
$\mathbf{p}_1 = r_{12}\mathbf{q}_1 = \sqrt{2}\cdot\frac{1}{\sqrt{2}}(-1,1)^T = (-1,1)^T$
$\mathbf{a}_2 - \mathbf{p}_1 = (3,5)^T - (-1,1)^T = (4,4)^T$
$r_{22} = \|\mathbf{a}_2 - \mathbf{p}_1\| = \|(4,4)^T\| = 4\sqrt{2}$
$\mathbf{q}_2 = \frac{1}{r_{22}}(\mathbf{a}_2 - \mathbf{p}_1) = \frac{1}{\sqrt{2}}(1,1)^T.$

(b) $r_{11} = \|\mathbf{a}_1\| = \|(2,1)^T\| = \sqrt{5}$
$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \frac{1}{\sqrt{5}}(2,1)^T$
$r_{12} = \mathbf{q}_1^T\mathbf{a}_2 = \frac{1}{\sqrt{5}}(2,1)(5,10)^T = \frac{20}{\sqrt{5}}$
$\mathbf{p}_1 = r_{12}\mathbf{q}_1 = \frac{20}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}(2,1)^T = (8,4)^T$
$\mathbf{a}_2 - \mathbf{p}_1 = (5,10)^T - (8,4)^T = (-3,6)^T$
$r_{22} = \|\mathbf{a}_2 - \mathbf{p}_1\| = \|(-3,6)^T\| = 3\sqrt{5}.$
$\mathbf{q}_2 = \frac{1}{r_{22}}(\mathbf{a}_2 - \mathbf{p}_1) = \frac{1}{\sqrt{5}}(-1,2)^T$

2 (a) $A = QR$ where $Q = [\mathbf{q}_1\ \mathbf{q}_2] = \frac{1}{\sqrt{2}}\begin{bmatrix}-1 & 1\\ 1 & 1\end{bmatrix}$ and $R = \begin{bmatrix}r_{11} & r_{12}\\ 0 & r_{22}\end{bmatrix} = \begin{bmatrix}\sqrt{2} & \sqrt{2}\\ 0 & 4\sqrt{2}\end{bmatrix}.$

(b) $A = QR$ where $Q = [\mathbf{q}_1\ \mathbf{q}_2] = \frac{1}{\sqrt{5}}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}$ and $R = \begin{bmatrix}r_{11} & r_{12}\\ 0 & r_{22}\end{bmatrix} = \begin{bmatrix}\sqrt{5} & 4\sqrt{5}\\ 0 & 3\sqrt{5}\end{bmatrix}.$

3 It is the same as Exercise 1 and now $A = \begin{bmatrix}1 & 4 & 1\\ 2 & 3 & 2\\ -2 & 2 & 1\end{bmatrix}.$

$r_{11} = \|\mathbf{a}_1\| = \|(1,2,-2)^T\| = 3$
$\mathbf{q}_1 = \frac{\mathbf{a}_1}{r_{11}} = \frac{1}{3}(1,2,-2)^T$
$r_{12} = \mathbf{q}_1^T\mathbf{a}_2 = \frac{1}{3}(1,2,-2)(4,3,2)^T = 2$
$\mathbf{p}_1 = r_{12}\mathbf{q}_1 = 2\cdot\frac{1}{3}(1,2,-2)^T = \frac{2}{3}(1,2,-2)^T$
$\mathbf{a}_2 - \mathbf{p}_1 = (4,3,2)^T - \frac{2}{3}(1,2,-2)^T = \frac{5}{3}(2,1,2)^T$
$r_{22} = \|\mathbf{a}_2 - \mathbf{p}_1\| = \|\frac{5}{3}(2,1,2)^T\| = \frac{5}{3}\|(2,1,2)^T\| = 5$
$\mathbf{q}_2 = \frac{1}{r_{22}}(\mathbf{a}_2 - \mathbf{p}_1) = \frac{1}{3}(2,1,2)^T$
$r_{13} = \mathbf{q}_1^T a_3 = \frac{1}{3}(1,2,-2)(1,2,1)^T = 1, \quad r_{23} = \mathbf{q}_2^T a_3 = \frac{1}{3}(2,1,2)(1,2,1)^T = 2$
$\mathbf{p}_2 = r_{13}\mathbf{q}_1 + r_{23}\mathbf{q}_2 = 1\cdot\frac{1}{3}(1,2,-2)^T + 2\cdot\frac{1}{3}(2,1,2)^T = \frac{1}{3}(5,4,2)^T$
$\mathbf{a}_3 - \mathbf{p}_2 = (1,2,1)^T - \frac{1}{3}(5,4,2)^T = \frac{1}{3}(-2,2,1)^T$
$r_{33} = \|\mathbf{a}_3 - \mathbf{p}_2\| = 1$
$\mathbf{q}_3 = \frac{1}{r_{33}}(a_3 - \mathbf{p}_2) = \frac{1}{3}(-2,2,1)^T$

Though the question doesn't ask,

$$A = QR = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$