Math 2660 Topics in Linear Algebra, Key

5.5

1,2,4,11,12,15,19

1 (a) $(1,0)(0,1)^T = 0$, (c) $(1,-1)(1,1)^T = 0$, (d) $\frac{1}{2}(\sqrt{3},1)\frac{1}{2}(-1,\sqrt{3})^T = 0$ so they form orthogonal basis of $\mathbb{R}^2$. (The answer of the book misses (c)).

2 (a) $\mathbf{u}_1^T\mathbf{u}_2 = \mathbf{u}_1^T\mathbf{u}_3 = \mathbf{u}_2^T\mathbf{u}_3 = 0$ and $\|\mathbf{u}_1\|_2 = \|\mathbf{u}_2\|_2 = \|\mathbf{u}_3\|_2 = 1$ so they are orthonormal. Since $\dim \mathbb{R}^3 = 3$, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_2\}$ is an orthonormal basis of $\mathbb{R}^3$.

(b) Let $\mathbf{x} = (1,1,1)^T$. Then

$$\mathbf{x} = (\mathbf{x}^T\mathbf{u}_1)\mathbf{u}_1 + (\mathbf{x}^T\mathbf{u}_2)\mathbf{u}_2 + (\mathbf{x}^T\mathbf{u}_3)\mathbf{u}_3 = -\frac{2}{3\sqrt{2}}\mathbf{u}_1 + \frac{5}{3}\mathbf{u}_2 + 0\cdot\mathbf{u}_3$$

Then $\|x\| =$

4 (a) $x_1^T x_2 = 0$ and $\|x_1\|^2 = \|x_2\|^2 = \cos^2\theta + \sin^2\theta = 1$. So x_1 and x_2 are orthonormal vectors. Since $\dim R^2 = 2$, $\{x_1, x_2\}$ is an orthonormal basis of $\mathbb{R}^2$.

(b) Let $y = (y_1, y_2)^T \in \mathbb{R}^2$. Then $y = (y^T x_1)x_1 + (y^T x_2)x_2 = (y_1\cos\theta + y_2\sin\theta)x_1 + (-y_1\sin\theta + y_2\cos\theta)x_2$.

(c) $\mathbf{x} - \mathbf{p} = (1,1,1,1)^T - \frac{1}{3}(4,1,1,0)^T = \frac{1}{3}(-1,2,2,3)^T$ and $\mathbf{p}^T(\mathbf{x}-\mathbf{p}) = \frac{1}{3}(4,1,1,0)\cdot\frac{1}{3}(-1,2,2,3)^T = 0$. Hence $\mathbf{p} \perp \mathbf{x} - \mathbf{p}$.

(d) $\|\mathbf{x}-\mathbf{p}\|_2 = \frac{1}{3}\sqrt{18} = \sqrt{2}$.
$\|\mathbf{p}\|_2 = \frac{1}{3}\sqrt{18} = \sqrt{2}$.
$\|\mathbf{x}\|_2 = 2$. Now $\mathbf{x} = \mathbf{p} + (\mathbf{x}-\mathbf{p})$ and $\mathbf{p} \perp \mathbf{x}-\mathbf{p}$. Pythagorean law is satisfied: $2^2 = (\sqrt{2})^2 + (\sqrt{2})^2$.

11 Since Q orthogonal implies $Q^T = Q^{-1}$ (p.259), then $Q = (Q^{-1})^T = (Q^T)^{-1}$. In other words, $QQ^T = I$, i.e., Q^T is orthogonal.

12 Let θ be the angle between $Q\mathbf{x}$ and $Q\mathbf{y}$ and let γ be the angle between $\mathbf{x}$ and $\mathbf{y}$. Notice that $\|P\mathbf{x}\|_2^2 = (P\mathbf{x})^T P\mathbf{x} = \mathbf{x}^T P^T P\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|_2^2$ and similarly $\|P\mathbf{y}\|_2^2 = \|\mathbf{x}\|_2^2$. Then

$$\cos\theta = (P\mathbf{x})^T P\mathbf{y}/(\|P\mathbf{x}\|_2\|P\mathbf{y}\|_2) = \mathbf{x}^T P^T P\mathbf{y}/(\|\mathbf{x}\|_2\|\mathbf{y}\|_2) = \mathbf{x}^T\mathbf{y}/(\|\mathbf{x}\|_2\|\mathbf{y}\|_2) = \cos\gamma.$$

So $\theta = \gamma$ as $\theta, \gamma \in [0, \pi]$.

15 Since $Q^TQ = I$, $(\det Q)^2 = \det Q^T \det Q = \det(Q^TQ) = \det I = 1$. So $|\det Q| = 1$, i.e., $\det Q$ is 1 or -1.

19 By Exercise 17, $UU^T = I$. Outer product form (p.78) of UU^T is $\mathbf{u}_1\mathbf{u}_1^T + \cdots + \mathbf{u}_n\mathbf{u}_n^T$ where $U = [\mathbf{u}_1 \ldots \mathbf{u}_n]$