Math 2660 Topics in Linear Algebra, Key

5.4

1-4

1 Let $\mathbf{x} = (-1,-1,1,1)^T$ and $\mathbf{y} = (1,1,5,-3)^T$. So

$$\mathbf{x}^T\mathbf{y} = (-1,-1,1,1)(1,1,5,-3)^T = 0.$$

Hence $\mathbf{x} \perp \mathbf{y}$. Moreover

$$\|\mathbf{x}\|_2 = ((-1)^2 + (-1)^2 + 1^2 + 1^2)^{1/2} = 2, \qquad \|\mathbf{y}\|_2 = (1^2 + 1^2 + 5^5 + (-3)^2)^{1/2} = 36^{1/2} = 6.$$

2 (a) $\cos\theta = \mathbf{x}^T\mathbf{y}/(\|\mathbf{x}\|_2\|\mathbf{y}\|_2) = (1,1,1,1)(8,2,2,0)^T/(2\cdot\sqrt{72}) = 6/\sqrt{72} = 1/\sqrt{2}$. So $\theta = 45^0$.

(b) $\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{y}^T\mathbf{y}}\mathbf{y} = \frac{(1,1,1,1)(8,2,2,0)^T}{72}(8,2,2,0)^T = \frac{1}{3}(4,1,1,0)^T$.

(c) $\mathbf{x}-\mathbf{p} = (1,1,1,1)^T - \frac{1}{3}(4,1,1,0)^T = \frac{1}{3}(-1,2,2,3)^T$ and $\mathbf{p}^T(\mathbf{x}-\mathbf{p}) = \frac{1}{3}(4,1,1,0)\cdot\frac{1}{3}(-1,2,2,3)^T = 0$. Hence $\mathbf{p} \perp \mathbf{x}-\mathbf{p}$.

(d) $\|\mathbf{x}-\mathbf{p}\|_2 = \frac{1}{3}\sqrt{18} = \sqrt{2}$.
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$\|\mathbf{x}\|_2 = 2$. Now $\mathbf{x} = \mathbf{p} + (\mathbf{x}-\mathbf{p})$ and $\mathbf{p} \perp \mathbf{x}-\mathbf{p}$. The Pythagorean law is satisfied: $2^2 = (\sqrt{2})^2 + (\sqrt{2})^2$.

3 Define $\langle\mathbf{x},\mathbf{y}\rangle = \frac{1}{4}x_1y_1 + \frac{1}{2}x_2y_2 + \frac{1}{4}x_3y_3$ by using the weight vector $\mathbf{w} = (\frac{1}{4},\frac{1}{2},\frac{1}{4})^T$. Let $\mathbf{x} = (1,1,1)^T$, $\mathbf{y} = (-5,1,3)^T$. Then

(a) $\langle\mathbf{x},\mathbf{y}\rangle = \frac{1}{4}\cdot 1(-5) + \frac{1}{2}\cdot 1(1) + \frac{1}{4}\cdot 1(3) = 0$. So $\mathbf{x}$ and $\mathbf{y}$ are perpendicular with respect to the inner product.

(b) $\|\mathbf{x}\| = (\frac{1}{4}\cdot 1^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{4}\cdot 1^2)^{1/2} = 1$ and $\|\mathbf{y}\| = (\frac{1}{4}\cdot(-5)^2 + \frac{1}{2}\cdot 1^2 + \frac{1}{4}\cdot 3^2)^{1/2} = \sqrt{36/4} = 3$.

4 (a) $\langle A,B\rangle = 1(-4) + 1(-3) + 3(1) + 2(1) + 0(3) + 1(-2) + 2(1) + 2(2) + 1(-2) = 0$.

(b) $\|A\|_F = (1^2 + 1^2 + 3^2 + 2^2 + 0^2 + 1^2 + 2^2 + 2^2 + 1^2)^{1/2} = \sqrt{25} = 5$.

(c) $\|B\|_F = (-4)^2 + (-3)^2 + 1^2 + 1^2 + 3^2 + (-2)^2 + 1^2 + 2^2 + (-2)^2 = \sqrt{49} = 7$.

(d) $A+B = \begin{bmatrix} -3 & 3 & 3 \\ 2 & 3 & 4 \\ 4 & -1 & -1 \end{bmatrix}$. So $\|A+B\|_F = ((-3)^2 + (-2)^2 + 4^2 + 3^2 + 3^2 + (-1)^2 + 3^2 + 4^2 + (-1)^2)^{1/2} = \sqrt{74}$. Note that in general $\|A+B\|_F \neq \|A\|_F + \|B\|_F$.