Math 2660 Topics in Linear Algebra, Key

5.3

1a,b,2,3,5

1 (a) $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix}$ so that $A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & -3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}$. Then the least square solution is given by $A^T A\mathbf{x} = A^T\mathbf{b}$ so that

$$\hat{\mathbf{x}} = \begin{bmatrix} 5 & -5 \\ -5 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{25} \begin{bmatrix} 10 & 5 \\ 5 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

(b) $A = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & -2 \end{bmatrix}$ so that $A^T A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}$. Then the least square solution is given by $A^T A\mathbf{x} = A^T\mathbf{b}$ so that

$$\hat{\mathbf{x}} = \begin{bmatrix} 6 & -1 \\ -1 & 6 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 10 \\ 5 \\ 20 \end{bmatrix} = \begin{bmatrix} 19/7 \\ -26/7 \end{bmatrix}.$$

2 (1b) (a) $\mathbf{p} = A\hat{\mathbf{x}} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 19/7 \\ -26/7 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -45 \\ 12 \\ 71 \end{bmatrix}$,

(b) $r(\hat{\mathbf{x}}) = \mathbf{b} - A\hat{\mathbf{x}} = \begin{bmatrix} 10 \\ 5 \\ 20 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} -45 \\ 12 \\ 71 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 115 \\ 23 \\ 69 \end{bmatrix}$.

(c) Set $\mathbf{r} = r(\hat{\mathbf{x}})$. Then $A^T\mathbf{r} = \begin{bmatrix} -1 & 2 & 2 \\ 1 & 1 & -2 \end{bmatrix} \frac{1}{7} \begin{bmatrix} 115 \\ 23 \\ 69 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Therefore $\mathbf{r} \in N(A^T)$.

3 (a) $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{bmatrix}$. Notice that $A^T A = \begin{bmatrix} 6 & 12 \\ 12 & 24 \end{bmatrix}$ is singular so that we cannot use $\hat{\mathbf{x}} = (A^T A)^{-1} A^T\mathbf{b}$. Now $A^T\mathbf{b} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 12 \end{bmatrix}$. Solve $A^T A\mathbf{x} = A^T\mathbf{b}$ to have

$$\left[\begin{array}{cc|c} 6 & 12 & 6 \\ 12 & 24 & 12 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

So $x_2 = t$ and $x_1 = 1 - 2t$. Thus the least square solution set is $\{(1 - 2t, t)^T : t \in \mathbb{R}\}$.

(b) $A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Notice that $A^T A = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 14 & 14 \\ 6 & 14 & 26 \end{bmatrix}$ is singular but checking the determinant so that we cannot use $\hat{\mathbf{x}} = (A^T A)^{-1} A^T \mathbf{b}$. Now $A^T \mathbf{b} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 3 & 2 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 26 \end{bmatrix}$.

Solve $A^T A \mathbf{x} = A^T \mathbf{b}$ to have

$$\left[\begin{array}{ccc|c} 3 & 0 & 6 & 6 \\ 0 & 14 & 14 & 14 \\ 6 & 14 & 26 & 26 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 1 & 1 \\ 6 & 0 & 0 & 0 \end{array}\right].$$

So $x_3 = t$, $x_2 = 1-t$, $x_2 = 2-2t$. Thus the least square solution set is $\{(2-2t, 1-t, t)^T : t \in \mathbb{R}\}$.

5 (a) $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}$ from the data and $A^T A = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$. Set $\mathbf{c} = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}$. So $A^T \mathbf{y} = \begin{bmatrix} 13 \\ 21 \end{bmatrix}$. Solving $A^T A \mathbf{c} = A^T \mathbf{y}$:

$$\left[\begin{array}{cc|c} 4 & 2 & 13 \\ 2 & 6 & 21 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{13}{4} \\ 0 & 1 & \frac{29}{10} \end{array}\right].$$

to have $c_1 = 2.9$ and $c_0 = 1.8$. So $y = 1.8 + 2.9x$.

(b) Plot the graph.