Math 2660 Topics in Linear Algebra, Key

5.2

1a,b,2,3,4

Warning: $R(A)$ denotes the range of A, i.e., it is the column space of A. So $R(A^T)$ is the row space of A. I made the changes on Key 3.6.

1 (a) $R(A^T) = \text{Row}\,(A)$: $\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \; R_2 - 2R_1 \; \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$. So $\{(3,4)\}$ is a basis for $R(A^T)$.

$R(A) = \text{Col}\,(A)$: $A^T = \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} \; \frac{1}{3}R_1 \; \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \; R_2 - 4R_1 \; \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. So $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$ is a basis for $\text{Col}\,(A)$.

$N(A)$: $\left[\begin{array}{cc|c} 3 & 4 & 0 \\ 6 & 8 & 0 \end{array}\right] \; R_2 - 2R_1 \; \left[\begin{array}{cc|c} 3 & 4 & 0 \\ 0 & 0 & 0 \end{array}\right]$. So $x_2 = t$ and $x_1 = -\frac{4}{3}t$. Then $\mathbf{x} = \frac{1}{3}t(-4,3)$ are the solution to $A\mathbf{x} = \mathbf{0}$. Thus $\{(-4,3)^T\}$ is a basis of $N(A)$.

$N(A^T)$: $\left[\begin{array}{cc|c} 3 & 6 & 0 \\ 4 & 8 & 0 \end{array}\right] \; \frac{1}{3}R_1 \; \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 4 & 8 & 0 \end{array}\right] \; R_2 - 4R_1 \; \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$. So $x_2 = t$ and $x_1 = -2t$. Thus $x = t(-2,1)$ are the solutions to $A^T\mathbf{x} = \mathbf{0}$. Thus $\{(-2,1)^T\}$ is a basis of $N(A^T)$.

(b) $R(A^T) = \text{Row}\,(A)$: $\begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 0 \end{bmatrix} \; R_2 - 2R_1 \; \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & -2 \end{bmatrix}$. So $\{(1,3,1), (0,-2,-2\}$ is a basis for $R(A^T)$.

$R(A) = \text{Col}\,(A)$: $A^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 1 & 0 \end{bmatrix} \; \begin{matrix} R_2 - 3R_1 \\ R_3 - R_1 \end{matrix} \; \begin{bmatrix} 1 & 2 \\ 0 & -2 \\ 0 & -2 \end{bmatrix} \; \begin{matrix} -\frac{1}{2}R_2 \\ R_3 - R_2 \end{matrix} \; \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. So $\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$ is a basis for $\text{Col}\,(A)$.

$N(A)$: $\left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 2 & 4 & 0 & 0 \end{array}\right] \; R_2 - 2R_1 \; \left[\begin{array}{ccc|c} 1 & 3 & 1 & 0 \\ 0 & -2 & -2 & 0 \end{array}\right]$. So $x_3 = t$, $x_2 = -t$ and $x_1 = 2t$. Then $\mathbf{x} = t(2,-1,1)$ are the solution. Thus $\{(2,-1,1)^T\}$ is a basis of $N(A)$.

$N(A^T)$: $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 4 & 0 \\ 1 & 0 & 0 \end{array}\right] \; \begin{matrix} R_2 - 3R_1 \\ R_3 - R_1 \end{matrix} \; \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{array}\right] \; \begin{matrix} -\frac{1}{2}R_2 \\ R_3 - R_2 \end{matrix} \; \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$. So the system has trivial solution only. Thus the empty set ϕ is the basis of $N(A^T)$.

2 (a) $S^\perp = \{(x_1,x_2,x_3)^T : (x_1,x_2,x_3)\cdot(1,-1,1) = 0\} = \{(x_1,x_2,x_3)^T : x_1 - x_2 + x_3 = 0\}$. Set $x_3 = t$, $x_2 = s$ so that $x_1 = s - t$. Thus $\mathbf{x} = s(1,1,0)^T + t(-1,0,1)^T$. Hence $\{(1,1,0)^T, (-1,0,1)^T\}$ is a basis.

(b) S corresponds to a line ℓ in $\mathbb{R}^3$ that passes through the origin and the point $(1,-1,1)$. So $S^\perp$ represents a plane in $\mathbb{R}^3$ that passes through the origin and is normal to ℓ.

3 (a) Let $S = \text{span}\,\{\mathbf{x}, \mathbf{y}\}$. Now $\mathbf{z} \in S^{\perp}$ means that $\mathbf{x}^T\mathbf{z} = \mathbf{y}^T\mathbf{z} = 0$. Put it in matrix form

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

i.e., $\mathbf{z} \in N(A)$ where $A = \begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$.

(b) Solving

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \quad R_2 - R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 \end{array}\right]$$

So $z_3 = t$, $z_2 = \frac{1}{3}t$ and $z_1 = -2z_2 - z_3 = -\frac{5}{3}t$, i.e., $\mathbf{z} = \frac{t}{3}(-5, 1, 3)^T$. So $\{(-5, 1, 3)^T\}$ is a basis of $S^{\perp}$.

4 Similar to Exercise 3, solving

$$\left[\begin{array}{cccc|c} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 3 & -2 & 0 \end{array}\right]$$

So $x_4 = t$, $z_3 = s$, $z_2 = -3s + 2t$ and $z_1 = 2s - t$, i.e., $\mathbf{z} = (2s - t, -3s + 2t, s, t)^T = s(2, -3, 1, 0)^T + t(-1, 2, 0, 1)^T$. So $\{(2, -3, 1, 0)^T, (-1, 2, 0, 1)^T\}$ is a basis of $S^{\perp}$.