Math 2660 Topics in Linear Algebra, Key

5.1

1,2,3a,b,4

1 (a) $\|v\| = (2^2+1^2+3^2)^{1/2} = \sqrt{14}$, $\|w\| = (6^2+3^2+9^2)^{1/2} = \sqrt{126}$. The two vectors are parallel so that $\theta = 0^0$. Or

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(2,1,3)(6,3,9)^T}{\sqrt{14}\sqrt{126}} = 1.$$

So $\theta = 0^0$.

(b) $\|v\| = (2^2+(-3)^2)^{1/2} = \sqrt{13}$, $\|w\| = (3^2+2^2)^{1/2} = \sqrt{13}$.

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(2,-3)(3,2)^T}{\sqrt{13}\sqrt{13}} = 0.$$

So $\theta = 90^0$.

(c) $\|v\| = (4^2+1^2)^{1/2} = \sqrt{17}$, $\|w\| = (3^2+2^2)^{1/2} = \sqrt{13}$.

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(4,1)(3,2)^T}{\sqrt{17}\sqrt{13}} = \frac{14}{\sqrt{221}}.$$

So $\theta \approx 10.65^0$ by calculator.

(d) $\|v\| = ((-2)^2+3^2+1^2)^{1/2} = \sqrt{14}$, $\|w\| = (1^2+2^2+4^2)^{1/2} = \sqrt{21}$.

$$\cos\theta = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{(-2,3,1)(1,2,4)^T}{\sqrt{14}\sqrt{21}} = \frac{4\sqrt{6}}{21}.$$

So $\theta \approx 62.19^0$ by calculator.

2 The scalar and vector projection of $\mathbf{v}$ onto $\mathbf{w}$ are given by the formula (p.214) respectively

$$\alpha = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{w}\|}, \qquad \mathbf{p} = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w}.$$

(a) $\alpha = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{w}\|} = \frac{(2,1,3)(6,3,9)^T}{\sqrt{6^2+3^2+9^2}} = \sqrt{14}$, $\mathbf{p} = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w} = \frac{(2,1,3)(6,3,9)^T}{6^2+3^2+9^2}(6,3,9)^T = (2,1,3)^T$.

(b) $\alpha = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{w}\|} = 0$, $\mathbf{p} = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w} = (0,0,0)^T = \mathbf{0}$.

(c) $\alpha = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{w}\|} = \frac{14\sqrt{13}}{13}$, $\mathbf{p} = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w} = \frac{1}{13}(42,28)^T$.

(d) $\alpha = \frac{\mathbf{v}^T\mathbf{w}}{\|\mathbf{w}\|} = \frac{8\sqrt{21}}{21}$, $\mathbf{p} = \frac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\mathbf{w} = \frac{1}{21}(8,16,32)^T$.

3 (a) $\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{y}}\mathbf{y} = (3,0)^T$, $\mathbf{x}-\mathbf{p} = (0,4)^T$. So $\mathbf{p}^T(\mathbf{x}-\mathbf{p}) = 3\cdot 0 + 0\cdot 4 = 0$.

(b) $\mathbf{p} = \frac{\mathbf{x}^T\mathbf{y}}{\mathbf{x}^T\mathbf{y}}\mathbf{y} = (4,4)^T$, $\mathbf{x}-\mathbf{p} = (-1,1)^T$. So $\mathbf{p}^T(\mathbf{x}-\mathbf{p}) = -4+4 = 0$.

4 If $\mathbf{x}$ and $\mathbf{y}$ are linearly independent and θ is the angle between the vectors, then $\mathbf{x}$ and $\mathbf{y}$ cannot be multiple to each other. Hence $|\cos\theta| < 1$. So

$$|\mathbf{x}^T\mathbf{y}| = \|\mathbf{x}\|\|\mathbf{y}\||\cos\theta| < 2\cdot 3 = 6.$$