Math 2660 Topics in Linear Algebra, Key

4.3

1a,b,c,2,4,7,11

1 (a) $L(x_1, x_2)^T = (-x_1, x_2)^T$. By inspection

$$\begin{aligned} L(\mathbf{u}_1) &= L(1,1)^T = (-1,1)^T = 0\mathbf{u}_1 + 1\mathbf{u}_2 \\ L(\mathbf{u}_2) &= L(-1,1)^T = (1,1)^T = 1\mathbf{u}_1 + 0\mathbf{u}_2 \end{aligned}$$

So the matrix of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $B = [L(\mathbf{u}_1)_U \; L(\mathbf{u}_2)_U] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

(b) $L(\mathbf{x}) = -\mathbf{x}$. So

$$\begin{aligned} L(\mathbf{u}_1) &= -\mathbf{u}_1 = -\mathbf{u}_1 + 0\mathbf{u}_2 \\ L(\mathbf{u}_2) &= -\mathbf{u}_2 = 0\mathbf{u}_1 - \mathbf{u}_2 \end{aligned}$$

So the matrix of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $B = [L(\mathbf{u}_1)_U \; L(\mathbf{u}_2)_U] = -I$.

(c) $L(\mathbf{x}) = L(x_1, x_2)^T = (x_2, x_1)$. So

$$\begin{aligned} L(\mathbf{u}_1) &= L(1,1)^T = \mathbf{u}_1 = \mathbf{u}_1 + 0\mathbf{u}_2 \\ L(\mathbf{u}_2) &= L(-1,1)^T = -\mathbf{u}_2 = 0\mathbf{u}_1 - \mathbf{u}_2 \end{aligned}$$

So the matrix of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $B = [L(\mathbf{u}_1)_U \; L(\mathbf{u}_2)_U] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

2 (a) The transition matrix from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$S = V^{-1}U = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = -\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}.$$

(b) From 1(a) the matrix of L with respect to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. So $SBS^{-1} = \begin{bmatrix} 1 & 0 \\ -4 & -1 \end{bmatrix}$ which is the matrix of L with respect to $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Direct verification.

4 (a) The transition matrix is given by $V = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) So the matrix representation of L with respect to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$ is

$$VAV^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where A is matrix representation of L with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ (given). This exercise illustrates that with respect to some suitable basis, the matrix representation of the same L is simpler.

7 If A is similar to B, then there exists a nonsingular matrix S such that $A = S^{-1}BS$. If B is similar to C, then there exists a nonsingular matrix R such that $B = R^{-1}CR$. It follows that

$$A = S^{-1}R^{-1}CRS = (RS)^{-1}C(RS)$$

where RS is nonsingular. So A is similar to C if A is similar to B and B is similar to C.

11 If $A = S^{-1}BS$, then $\det A = \det S^{-1} \det B \det S = \frac{1}{\det S} \det B \det S = \det B$.