Math 2660 Topics in Linear Algebra, Key

4.2

2-4,6-7

2 (a) $L(\mathbf{e}_1) = L(1,0,0)^T = (1,0)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (1,0)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (0,0)^T$. So the matrix representation with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ is

$$[L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) $L(\mathbf{e}_1) = L(1,0,0)^T = (1,0)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (0,1)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (0,0)^T$. So the matrix representation with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ is

$$[L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) $L(\mathbf{e}_1) = L(1,0,0)^T = (-1,0)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (1,-1)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (0,1)^T$. So the matrix representation with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ and $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ is

$$[L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

3 A is the matrix representation of L with respect to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$.

(a) $L(\mathbf{e}_1) = L(1,0,0)^T = (0,0,1)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (0,1,0)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (1,0,0)^T$. So the matrix representation A with respect to the bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ is

$$[L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(b) $L(\mathbf{e}_1) = L(1,0,0)^T = (1,1,1)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (0,1,1)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (0,0,1)^T$. So the matrix representation A with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ is

$$[L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(c) $L(\mathbf{e}_1) = L(1,0,0)^T = (0,3,2)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (0,1,0)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (2,0,-1)^T$. So the matrix representation A with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ is

$$[L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & -1 \end{bmatrix}.$$

4 $L(\mathbf{e}_1) = L(1,0,0)^T = (2,-1,-1)^T$, $L(\mathbf{e}_2) = (L(0,1,0)^T = (-1,2,-1)^T$, $L(\mathbf{e}_3) = L(0,0,1)^T = (-1,-1,2)^T$. So the matrix representation A with respect to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$ is

$$A = [L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(a) $L(x) = Ax = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$

(b) $L(x) = Ax = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}.$

(c) $L(x) = Ax = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -15 \\ 9 \\ 6 \end{bmatrix}.$

6 The matrix representation of L from the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $\mathbb{R}^2$ to $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is given by

$$A = [[L(\mathbf{e}_1)]_B\ [L(\mathbf{e}_2)]_B].$$

Now $L(\mathbf{e}_1) = L(1,0)^T = \mathbf{b}_1 + \mathbf{b}_3$, $L(\mathbf{e}_2) = L(0,1)^T = \mathbf{b}_2 + \mathbf{b}_3$. So $[L(\mathbf{e}_1)]_B = (1,0,1)^T$ and $[L(\mathbf{e}_2)]_B = (0,1,1)^T$. Then

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

7 (a) By the definition of $\mathcal{I}$,

$$\begin{aligned} \mathcal{I}(e_1) &= \mathbf{e}_1 = c_{11}\mathbf{y}_1 + c_{12}\mathbf{y}_2 + c_{13}\mathbf{y}_3 \\ \mathcal{I}(e_2) &= \mathbf{e}_2 = c_{21}\mathbf{y}_1 + c_{22}\mathbf{y}_2 + c_{23}\mathbf{y}_3 \\ \mathcal{I}(e_2) &= \mathbf{e}_3 = c_{31}\mathbf{y}_1 + c_{32}\mathbf{y}_2 + c_{33}\mathbf{y}_3 \end{aligned}$$

Solving $\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$ to have

$$\begin{aligned} \mathcal{I}(e_1) &= = 0\mathbf{y}_1 + 0\mathbf{y}_2 + 1\mathbf{y}_3 \\ \mathcal{I}(e_2) &= = 0\mathbf{y}_1 + 1\mathbf{y}_2 - 1\mathbf{y}_3 \\ \mathcal{I}(e_2) &= = 1\mathbf{y}_1 - 1\mathbf{y}_2 + 0\mathbf{y}_3 \end{aligned}$$

So the coordinate vectors of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with respect to the ordered basis $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ are

$$[\mathbf{e}_1]_Y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, [\mathbf{e}_2]_Y = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, [\mathbf{e}_3]_Y = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

(b) The matrix A is the change of basis matrix from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$. So

$$A = [[\mathbf{e}_1]_Y\ [\mathbf{e}_2]_Y\ [\mathbf{e}_3]_Y] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}.$$