Math 2660 Topics in Linear Algebra, Key

4.1

1-5,7

1 (a) reflection about x_2 axis.

(b) reflection about the origin.

(c) reflection about the line $x_2 = x_1$.

(d) the length of the vector is halved.

(e) projection onto x_2 axis.

2 Express $\mathbf{x} = (x_1, x_2)$ in terms of polar coordinates: $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ where $r = (x_1^2 + x_2^2)^{1/2}$ and θ is the angle between $\mathbf{x}$ and $\mathbf{e}_1$. Now

$$\begin{aligned} L(\mathbf{x}) &= (r\cos\theta\cos\alpha - r\sin\theta\sin\alpha, r\cos\theta\sin\alpha + r\sin\theta\cos\alpha)^T \\ &= (r\cos(\theta+\alpha), r\sin(\theta+\alpha))^T \end{aligned}$$

So $(x_1, x_2) = (r\cos\theta, x_2 = r\sin\theta)^T$ is mapped to $(r\cos(\theta+\alpha), r\sin(\theta+\alpha))^T$ by L. The linear transformation L has the effect of rotating a vector by an α in the counterclockwise direction.

3 If $\alpha \neq 0$, then

$$L(\alpha\mathbf{x}) = \alpha\mathbf{x} + a \neq \alpha\mathbf{x} + \alpha\mathbf{a} = \alpha L(\mathbf{x}).$$

So the scalar multiplication property fails. Indeed addition property also fails since

$$\begin{aligned} L(\mathbf{x}+\mathbf{y}) &= \mathbf{x}+\mathbf{y}+\mathbf{a} \\ L(\mathbf{x})+L(\mathbf{y}) &= \mathbf{x}+\mathbf{a}+\mathbf{y}+\mathbf{a} = \mathbf{x}+\mathbf{y}+2\mathbf{a} \end{aligned}$$

4 Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 7 \\ 5 \end{bmatrix}.$$

Express $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2$. Solving the system

$$\left[\begin{array}{cc|c} 1 & 1 & 7 \\ 2 & -1 & 5 \end{array}\right] \quad R_2 - 2R_1 \quad \left[\begin{array}{cc|c} 1 & 1 & 7 \\ 0 & -3 & -9 \end{array}\right] \rightarrow (c_1, c_2) = (4, 3)$$

Hence $\mathbf{x} = 4\mathbf{u}_1 + 3\mathbf{u}_2$. Thus

$$T(\mathbf{x}) = 4T(\mathbf{u}_1) + 3T(\mathbf{u}_2) = 4(-2,3)^T + 3(5,2)^T = (7,18)^T$$

since $T(\mathbf{u}_1) = (-2,3)^T$ and $T(\mathbf{u}_2) = (5,2)^T$.

5 (a) L is a linear transformation because

$$\begin{aligned} L(\alpha\mathbf{x}+\beta\mathbf{y}) &= L(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, \alpha x_3 + \beta y_3)^T \\ &= (\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2)^T \quad \text{by the definition of } L \\ &= \alpha(x_1, x_2)^T + \beta(y_1, y_2)^T \\ &= \alpha L(\mathbf{x}) + \beta L(\mathbf{y}) \quad \text{by the definition of } L \end{aligned}$$

(b) L is a linear transformation because

$$\begin{aligned} L(\alpha\mathbf{x}+\beta\mathbf{y}) &= (0,0)^T \quad \text{by the definition of } L \\ &= \alpha L(\mathbf{x})+\beta L(\mathbf{y}) \quad \text{by the definition of } L \end{aligned}$$

(c) It is not since $2L(1,1)^T = 2(2,1)^T$ but $L(2,2) = (3,2)^T$, i.e., $L(2,2)^T \neq 2L(1,1)^T$. So it is not a linear transformation. Or simply say $L(0,0)^T = (1,0)^T \neq (0,0)^T$ since a linear transformation L must satisfy $T(\mathbf{0}) = \mathbf{0}$.

(d) L is a linear transformation because

$$\begin{aligned} L(\alpha\mathbf{x}+\beta\mathbf{y}) &= L(\alpha x_1+\beta y_1, \alpha x_2+\beta y_2, \alpha x_3+\beta y_3)^T \\ &= (\alpha x_3+\beta y_3, \alpha x_1+\beta y_1+\alpha x_2+\beta y_2)^T \quad \text{by the definition of } L \\ &= \alpha(x_3, x_1+x_2)^T+\beta(y_3, y_1+y_2)^T \\ &= \alpha L(\mathbf{x})+\beta L(\mathbf{y}) \quad \text{by the definition of } L \end{aligned}$$

7 (a) $L(\alpha A+\beta B) = 2(\alpha A+\beta B) = \alpha(2A)+\beta(2B) = \alpha L(A)+\beta L(B)$. So L is a linear transformation.

(b) $L(\alpha A+\beta B) = (\alpha A+\beta B)^T = \alpha A^T+\beta B^T = \alpha L(A)+\beta L(B)$. So L is a linear transformation.

(c) $L(0) = 0+I = I \neq 0$. So L is not a linear transformation.

(d) $L(\alpha A+\beta B) = (\alpha A+\beta B)-(\alpha A+\beta B)^T = (\alpha A+\beta B)-(\alpha A^T+\beta B^T) = \alpha A-\alpha A^T+\beta B-\beta B^T = \alpha L(A)+\beta L(B)$. So L is a linear transformation.