Math 2660 Topics in Linear Algebra, Key

3.6

1,2,4,6,8

We will denote by Row (A) the row space of A and Col (A) the column space of A. Notice that Col (A) = Row (A^T).

1 (a) Row (A): $A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 7 & 8 \end{bmatrix} \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -5 & 0 \end{bmatrix} \begin{matrix} \\ -\frac{1}{5}R_2 \\ R_3 - R_2 \end{matrix} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

So $\{(1,3,2),(0,1,0)\}$ is a basis for Row (A), the row space of A.

Remark: One may get the reduced row echelon of A which is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and it is okay. The basis is $\{(1,0,2),(0,1,0)\}$. The answer is also correct since basis is not unique. Same remark applies for the other parts.

So rank $A = 2$.

Col (A): $A^T = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 2 & 4 & 8 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} -\frac{1}{5}R_2 \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

The take the transpose back.

Remark: The whole process is to do column operations on the columns. So $\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \}$ is a basis for Col (A), the column space of A.

$N(A)$: $[A|0] = \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 2 & 1 & 4 & 0 \\ 4 & 7 & 8 & 0 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -5 & 0 & 0 \end{array}\right] \begin{matrix} \\ -\frac{1}{5}R_2 \\ R_3 - R_2 \end{matrix} \left[\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$.

So $x_3 = t$, $x_2 = 0$, $x_1 = -2t$, i.e., $N(A) = \text{span } (-2,0,1)^t$ and $\{(-2,0,1)^t\}$ is a basis for $N(A)$. So the nullity of A is 1. It matches with rank A + nullity$A = n$ where A is $m \times n$.

(b) Row (A): $A = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \begin{matrix} \\ 3R_2 + R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 7 & 1 & -2 \end{bmatrix} \begin{matrix} \\ \\ R_3 - R_2 \end{matrix} \begin{bmatrix} -3 & 1 & 3 & 4 \\ 0 & 7 & 0 & -2 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

So $\{(-3,1,3,4),(0,7,0,-2),(0,0,1,0)\}$ is a basis for Row (A), the row space of A.

So rank $A = 3$.

Col (A): $A^T = \begin{bmatrix} -3 & 1 & 3 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 4 & 2 \end{bmatrix} \begin{matrix} 3R_2 + R_1 \\ R_3 + R_1 \\ 3R_4 + 4R_1 \end{matrix} \begin{bmatrix} -3 & 1 & -3 \\ 0 & 7 & 21 \\ 0 & 0 & 1 \\ 0 & -2 & -12 \end{bmatrix} \to \begin{bmatrix} -3 & 1 & -3 \\ 0 & 7 & 21 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

So $\{ \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 21 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$ is a basis for Col (A), the column space of A.

$N(A)$: $[A|0] = \left[\begin{array}{cccc|c} -3 & 1 & 3 & 4 & 0 \\ 1 & 2 & -1 & -2 & 0 \\ -3 & 8 & 4 & 2 & 0 \end{array}\right] \begin{matrix} \\ R_2 - 2R_1 \\ R_3 - 4R_1 \end{matrix} \left[\begin{array}{cccc|c} -3 & 1 & 3 & 4 & 0 \\ 0 & 7 & 0 & -2 & 0 \\ 0 & 7 & 1 & -2 & 0 \end{array}\right]$

$$\begin{array}{r} \\ -\frac{1}{5}R_2 \\ R_3 - R_2 \end{array} \left[\begin{array}{cccc|c} -3 & 1 & 3 & 4 & 0 \\ 0 & 7 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

So $x_4 = t$, $x_3 = 0$, $x_2 = -\frac{2}{7}t$, $x_1 = -\frac{1}{3}(x_2 + 4x_4) = \frac{1}{3}(-\frac{2}{7}t + 4t) = \frac{26}{21}t$, i.e.,

$$x = t(\frac{26}{21}, -\frac{2}{7}, 0, 1) = \frac{t}{21}(26, -6, 0, 21)$$

and thus $N(A) = \text{span }(26, -6, 0, 21)^t$ and $\{(26, -6, 0, 21)^t\}$ is a basis for $N(A)$. So the nullity of A is 1. It matches with rank $A + \text{nullity}A = n$ where A is $m \times n$.

(c) Row (A): $A = \begin{bmatrix} 1 & 3 & -2 & 1 \\ 2 & 1 & 3 & 2 \\ 3 & 4 & 5 & 6 \end{bmatrix} \begin{array}{r} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & -8 & 11 & 3 \end{bmatrix} \begin{array}{r} \\ \\ 5R_3 - 8R_2 \end{array} \begin{bmatrix} 1 & 3 & -2 & 1 \\ 0 & -5 & 7 & 0 \\ 0 & 0 & -1 & 15 \end{bmatrix}.$

So $\{(1, 3, -2, 1), (0, -5, 7, 0), (0, 0, -1, 15)\}$ is a basis for Row (A), the row space of A. Thus rank $A = 3$.

Col (A): $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 4 \\ -2 & 3 & 5 \\ 1 & 2 & 6 \end{bmatrix} \begin{array}{r} \\ R_2 - 3R_1 \\ R_3 + 2R_1 \\ R_4 - R_1 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 7 & 11 \\ 0 & 0 & 3 \end{bmatrix} \begin{array}{r} \\ -\frac{1}{5}R_2 \\ \\ \frac{1}{3}R_4 \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 7 & 11 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{array}{r} \\ \\ R_3 - 7R_2 \\ \\ \end{array} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. So $\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for Col (A), the column space of A.

$N(A)$: $A = \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 2 & 1 & 3 & 2 & 0 \\ 3 & 4 & 5 & 6 & 0 \end{array}\right] \begin{array}{r} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 0 & -5 & 7 & 0 & 0 \\ 0 & -8 & 11 & 3 & 0 \end{array}\right] \begin{array}{r} \\ \\ 5R_3 - 8R_2 \end{array}$

$\left[\begin{array}{cccc|c} 1 & 3 & -2 & 1 & 0 \\ 0 & -5 & 7 & 0 & 0 \\ 0 & 0 & -1 & 15 & 0 \end{array}\right].$

So $x_4 = t$, $x_3 = 15t$, $x_2 = 21t$ and $x_1 = -31t$, i.e., $x = t(-31, 21, 15, 1)$. So $N(A) = \text{span }(-31, 21, 15, 1)^t$ and $\dim N(A) = 1$. It matches with rank $A + \text{nullity}A = n$ where A is $m \times n$.

4 (a) $\left[\begin{array}{cc|c} 1 & 2 & 4 \\ 2 & 4 & 8 \end{array}\right] \begin{array}{r} \\ R_2 - 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & 0 \end{array}\right]$. So it is consistent. Thus b is in the column space of A.

(b) $\left[\begin{array}{cc|c} 3 & 6 & 1 \\ 1 & 2 & 1 \end{array}\right] \begin{array}{r} \\ R_2 - 2R_1 \end{array} \left[\begin{array}{cc|c} 1 & 2 & 4 \\ 0 & 0 & -\frac{2}{3} \end{array}\right]$. So it is inconsistent. Thus b is not in the column space of A.

Similarly (c), (f), (e) are consistent so b is in the column space of A. (d) is inconsistent.

6 There will be infinitely many solutions. The condition that $\mathbf{b}$ is in the column space of A guarantees that the system is consistent. Let $\mathbf{x}_p$ be a solution to $A\mathbf{x} = \mathbf{b}$, i.e., $A\mathbf{x}_p = \mathbf{b}$. If the column vectors are linearly dependent, there are infinitely many solutions $\mathbf{x}_0$ to $A\mathbf{x} = 0$, i.e., $A\mathbf{x}_0 = 0$. Then there are infinitely many $\mathbf{x}_p + \mathbf{x}_0$ and $A(\mathbf{x}_p + \mathbf{x}_0) = 0$. In other words there are infinitely many solution to $A\mathbf{x} = \mathbf{b}$.

8 $\dim N(A) + \text{rank } A = 5$ by Theorem 3.6.5 since A is 6×5. So $\text{rank } A = 5 - \dim N(A) = 5 - 2 = 3$.

$\dim N(B) = 5 - \text{rank } B = 5 - 4 = 1$.