Math 2660 Topics in Linear Algebra, Key

3.5

1-3, 5-9

1 The transition matrix S from $\{\mathbf{u}_1, \mathbf{u}_2\}$ to $\{\mathbf{e}_1, \mathbf{e}_2\}$ is given by

(a) $S = [\mathbf{u}_1\mathbf{u}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$

(b) $S = [\mathbf{u}_1\mathbf{u}_2] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$

(c) $S = [\mathbf{u}_1\mathbf{u}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

2 The transition matrix S from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{u}_1, \mathbf{u}_2\}$ is given by the inverse of those in Exercise 1.

(a) $S^{-1} = [\mathbf{u}_1\mathbf{u}_2]^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$

(b) $S^{-1} = [\mathbf{u}_1\mathbf{u}_2]^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}.$

(c) $S^{-1} = [\mathbf{u}_1\mathbf{u}_2]^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$

3 (a) $S = U^{-1}V = [\mathbf{u}_1\mathbf{u}_2]^{-1}[\mathbf{v}_1\mathbf{v}_2] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 5 & 7 \\ -1 & -1 \end{bmatrix}.$

(b) $S = U^{-1}V = [\mathbf{u}_1\mathbf{u}_2]^{-1}[\mathbf{v}_1\mathbf{v}_2] = \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 11 & 14 \\ -4 & -5 \end{bmatrix}.$

(c) $S = U^{-1}V = [\mathbf{u}_1\mathbf{u}_2]^{-1}[\mathbf{v}_1\mathbf{v}_2] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}.$

5 (a) The transition matrix from $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $U = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is

$$S = [\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3]^{-1} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 2 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$

(b) (i) Let $\mathbf{x} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$. Then $[\mathbf{x}]_U = S[\mathbf{x}]_E = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$

(ii) Let $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Then $[\mathbf{x}]_U = S[\mathbf{x}]_E = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$

(iii) Let $\mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}$. Then $[\mathbf{x}]_U = S[\mathbf{x}]_E = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}.$

7 Let $V = [\mathbf{v}_1\mathbf{v}_2]$, $W = [\mathbf{w}_1\mathbf{w}_2]$. Then the transition matrix from $\{\mathbf{w}_1, \mathbf{w}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ is

$$S = V^{-1}W \Rightarrow W = VS \Rightarrow W = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 9 & 4 \end{bmatrix}.$$

So $\mathbf{w}_1 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$.

8 $S = U^{-1}V \Rightarrow US = V \Rightarrow U = VS^{-1} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 5 \end{bmatrix}.$

So $\mathbf{u}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

9 (a)

$$\begin{aligned} 2x - 1 &= 2 \cdot x - 1 \cdot 1 \\ 2x + 1 &= 2 \cdot x + 1 \cdot 1 \end{aligned}$$

so that the transition matrix from $\{2x - 1, 2x + 1\}$ to $\{x, 1\}$ is $S = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$.

(b) The transition matrix from $\{x, 1\}$ to $\{2x - 1, 2x + 1\}$ is $S^{-1} = \frac{1}{4} \begin{bmatrix} 1 & -2 \\ 1 & 2 \end{bmatrix}$.