

## Math 2660 Topics in Linear Algebra, Key

### 3.4

1a,b,c,2a,c,e,3,4,5,7,8

- 1 (a) Yes, since the two vectors are linearly independent in  $\mathbb{R}^2$  (see Theorem 3.4.3).  
(b) No, since the vectors are linearly dependent (see Corollary 3.4.2).  
(c) No, since the three vectors are linearly dependent (see Corollary 3.4.2).
- 2 (a) Yes, since the three vectors are linearly independent (see Theorem 3.4.3).  
(c) Yes, since the three vectors are linearly independent (see Theorem 3.4.3).  
(e) No, since there are only two vectors (see Corollary 3.4.2).
- 3 (a)  $\begin{vmatrix} 2 & 4 \\ 1 & 3 \end{vmatrix} = 2 \neq 0$ . So the vectors  $\mathbf{x}_1, \mathbf{x}_2$  are linearly independent and span  $\mathbb{R}^2$  (since  $\dim \mathbb{R}^2 = 2$ ).  
(b) Use Theorem 3.4.1.  
(c)  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \mathbb{R}^3$ . So the dimension is 2.
- 4 Notice that  $\mathbf{x}_3 = 2\mathbf{x}_2$  and  $\mathbf{x}_2 = -\mathbf{x}_1$  so that  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \text{span}\{\mathbf{x}_1\}$  so the dimension is 1.

5 (a)  $\begin{vmatrix} 2 & 3 & 2 \\ 1 & -1 & 6 \\ 3 & 4 & 4 \end{vmatrix} = 0$  (check!) so that the vectors are linearly dependent.

(b) Set  $c_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ . Solving the system

$$\left[ \begin{array}{cc|c} 2 & 3 & 0 \\ 1 & -1 & 0 \\ 3 & 4 & 0 \end{array} \right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - \frac{3}{2}R_2 \end{array} \left[ \begin{array}{cc|c} 2 & 3 & 0 \\ 0 & -\frac{5}{2} & 0 \\ 0 & -\frac{1}{2} & 0 \end{array} \right].$$

So  $c_1 = c_2 = 0$ . Thus  $\mathbf{x}_1, \mathbf{x}_2$  are linearly independent.

- (c)  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\} = \text{span}\{\mathbf{x}_1, \mathbf{x}_2\}$  So the dimension of  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is 2.
  - (d)  $\text{span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$  is a plane passing through the origin.
- 7 Let  $S = \{(a + b, a - b + 2c, b, c)^T : a, b, c \in \mathbb{R}\}$ . Notice that

$$(a + b, a - b + 2c, b, c)^T = a(1, 1, 0, 0)^T + b(1, -1, 1, 0)^T + c(0, 2, 0, 1)^T.$$

So the three vectors  $(1, 1, 0, 0)^T, (1, -1, 1, 0)^T, (0, 2, 0, 1)^T$  span  $S$ . Moreover, they are linearly independent since if  $a(1, 1, 0, 0)^T + b(1, -1, 1, 0)^T + c(0, 2, 0, 1)^T = 0$ , i.e.,  $(a + b, a - b + 2c, b, c)^T = 0$ , then  $a = b = c = 0$ . So the dimension of  $S$  is 3.

- 8 (a) No, because we only have two vectors  $\mathbf{x}_1, \mathbf{x}_2$ .  
(b)  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  need to be linearly independent in order to span  $\mathbb{R}^3$ .

(c) Set  $\mathbf{x}_3 = \mathbf{e}_1$ . Then  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$  are linearly independent since  $\begin{vmatrix} 1 & 3 & 1 \\ 1 & -1 & 0 \\ 1 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 4 \end{vmatrix} = 5 \neq 0$ .

0. Indeed you could have many choices.