Math 2660 Topics in Linear Algebra, Key

3.3

1a,b,c,2,a,c,e,3a,c,e,4,5,6,12,13,16,17

1 (a) Set $c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Solving the system

$$\left[\begin{array}{cc|c} 2 & 3 & 0 \\ 1 & 2 & 0 \end{array}\right] \quad R_2 - \tfrac{1}{2}R_1 \quad \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & \tfrac{1}{2} & 0 \end{array}\right].$$

So $c_1 = c_2 = 0$ and thus the vectors are linearly independent. We can get the same conclusion by using Theorem 3.3.1 since det $\begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \neq 0$.

(b) Since $\begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} = 0$, by Theorem 3.3.1, the vectors are linearly dependent.

(c) Set $c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Solving the system

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 1 & 3 & 4 & 0 \end{array}\right] \quad R_2 + \tfrac{1}{2}R_1 \quad \left[\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 0 & \tfrac{7}{2} & 5 & 0 \end{array}\right].$$

So there are infinitely many solutions. Thus the vectors are linearly dependent. Theorem 3.3.1 does not apply here.

2 (a) Set $c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving the system

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \quad R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right].$$

So $c_1 = c_2 = c_3 = 0$. Thus the vectors are linearly independent. Or use Theorem 3.3.1 to check the determinant.

(c) $\begin{vmatrix} 2 & 3 & 2 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix} \; R_1 - R_2 \; = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ -2 & -2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$. So the vectors are linearly independent by Theorem 3.3.1.

(e) Set $c_1 \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving the system

$$\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right] \quad \begin{array}{c} R_1 - R_2 \\ R_3 - 3R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 0 \end{array}\right].$$

So $c_1 = c_2 = 0$ and hence the vectors are linearly independent. Theorem 3.3.1 does not apply here.

3 (a) The span of the vectors is $\mathbb{R}^3$.

(b) The span of the vectors is $\mathbb{R}^3$.

(c) The span of the vectors is a plane passing through the origin.

4 Now "vectors" are matrices since the vector space is $\mathbb{R}^{2\times 2}$ and the zero matrix is the identity element.

(a) Set $c_1 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $c_1 = 0$ by equating the $(1,1)$ entries of both sides, and $c_2 = 0$ by equating the $(1,2)$ entries of both sides. Thus they are linearly independent.

(b) Set $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $\begin{bmatrix} c_1 & c_2 \\ c_3 & c_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $c_1 = c_2 = c_3 = 0$. Thus they are linearly independent.

(c) Set $c_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So $\begin{bmatrix} c_1 & c_2 + 3c_3 \\ 0 & c_1 + 2c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $c_1 = c_2 = c_3 = 0$. Thus they are linearly independent.

5 (a) We may not have a linearly independent set if we add a vector $\mathbf{x}_{k+1}$ to the linearly independent set $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$. For example, $\{\mathbf{e}_1, \mathbf{e}_2\}$ is a linearly independent set but $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2\}$ is linearly dependent after we add $\mathbf{e}_1 + \mathbf{e}_2$ into the set.

(b) Yes, if we delete a vector from a linearly independent set, say, $\mathbf{x}_k$, then $\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$ is a linearly independent set. Set $c_1\mathbf{x}_1 + \cdots + c_{k-1}\mathbf{x}_{k-1} = \mathbf{0}$. We try to show that $c_1 = \cdots = c_{k-1} = 0$. The equation can be interpreted as $c_1\mathbf{x}_1 + \cdots + c_{k-1}\mathbf{x}_{k-1} + 0\mathbf{x}_k = \mathbf{0}$. But the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are linearly independent so that $c_1 = \cdots c_{k-1} = 0$. Hence $\{\mathbf{x}_1, \ldots, \mathbf{x}_{k-1}\}$ is a linearly independent set.

6 (a) Now the "vectors" are polynomials. Set $c_1(1) + c_2x^2 + c_3(x^2 - 2) = 0$, the zero polynomial. By considering the coefficients of x^2, x and the constant term on both sides: x^2: $c_2 + c_3 = 0$, x: nothing, constant term: $c_1 - 2c_3 = 0$. So c_2 is a parameter and thus we have infinitely many solution. Hence they are linearly dependent. Or by inspection: $x^2 - 2 = x^2 - 2(1)$, i.e., the last vector is a linear combination of the first two.

(c) Set $c_1(x+2) + c_2(x+1) + c_3(x^2 - 1) = 0$, the zero polynomial. By considering the coefficients of x^2, x and the constant term on both sides: x^2: $c_3 = 0$, x: $c_1 + c_2 = 0$, constant term: $2c_1 + c_2 - c_3 = 0$. So $c_1 = c_2 = c_3 = 0$. Hence they are linearly independent.

12 $\mathbf{v}_1, \mathbf{v}_2$ are linearly dependent means there are scalars c_1, c_2, not all zeros, such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = 0$. For example, if $c_1 \neq 0$, then $\mathbf{v}_1 = -\frac{c_2}{c_1}\mathbf{v}_2$, i.e., $\mathbf{v}_1$ is a multiple of $\mathbf{v}_2$. Similar for $c_2 \neq 0$.

13 It is similar to Ex 5 (b).

16 Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a spanning set for V, $\mathbf{v}$ is spanned by them, i.e., $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ for some $c_1, \ldots, c_n$. Then $c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n + (-1)\mathbf{v}_n = \mathbf{0}$. Hence $\{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{v}\}$ is linearly dependent.

17 Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be linearly independent. Notice that $\mathbf{v}_1$ cannot be spanned by $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$. It is because if $\mathbf{v}_1 = c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ were true, i.e., $-\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{0}$. This would imply $-1 = c_2 = \cdots = c_n = 0$, contradiction. Hence $\{\mathbf{v}_2, \ldots, \mathbf{v}_n\}$ cannot span V.