Math 2660 Topics in Linear Algebra, Key

3.2

1a,b,c,d,2a,b,c,3a,b,c,d,4a,b,9a,c,10a,b,e,11,12

1 (a) $S = \{\mathbf{x} = (x_1, x_2)^T : x_1 + x_2 = 0\}$. (i) $\mathbf{0} \in S$ since $0 + 0 = 0$ so S is not empty. (ii) If $\mathbf{x}, \mathbf{y} \in S$, i.e, $x_1 + x_2 = 0 = y_1 + y_2$, then $\mathbf{x} + \mathbf{y} \in S$ since $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)^T$ and $x_1 + y_1 + x_2 + y_2 = x_1 + x_2 + y_1 + y_2 = 0$. (iii) If $\mathbf{x} \in S$ and $\alpha \in \mathbb{R}$, i.e, $x_1 + x_2 = 0$, then $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2) \in S$ since $\alpha x_1 + \alpha x_2 = \alpha(x_1 + x_2) = \alpha \cdot 0 = 0$.
So S is a subspace of $\mathbb{R}^2$.

(b) $S = \{\mathbf{x} = (x_1, x_2)^T : x_1 x_2 = 0\}$ is not a subspace of $\mathbb{R}^2$ since $(1, 0)^T, (0, 1)^T \in S$ but their sum $(1, 1)^T \notin S$. Remark: S is closed under scalar multiplication and S is nonempty since $\mathbf{0} \in S$.

(c) $S = \{\mathbf{x} = (x_1, x_2)^T : x_1 = 3x_2\}$. (i) $\mathbf{0} \in S$ since $0 = 3 \cdot 0$ so S is not empty. (ii) If $\mathbf{x}, \mathbf{y} \in S$, i.e, $x_1 = 3x_2$, $y_1 = 3y_2$, then $\mathbf{x} + \mathbf{y} \in S$ since $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)^T$ and $x_1 + y_1 = 3x_2 + 3y_2 = 3(x_2 + y_2)$. (iii) If $\mathbf{x} \in S$ and $\alpha \in \mathbb{R}$, i.e, $x_1 = 3x_2$, then $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2) \in S$ since $\alpha x_1 = \alpha(3x_2) = 3(\alpha x_2)$.
So S is a subspace of $\mathbb{R}^2$.

(d) $S = \{\mathbf{x} = (x_1, x_2)^T : |x_1| = |x_2|\}$ is not a subspace of $\mathbb{R}^2$ since $(1, -1)^T, (1, 1)^T \in S$ but their sum $(2, 0)^T \notin S$ as $|2| = 2 \neq 0 = |0|$.

2 (a) $S = \{\mathbf{x} = (x_1, x_2, x_3)^T : x_1 + x_3 = 1\}$ is not a subspace of $\mathbb{R}^3$ since $(1, 0, 0)^T, (0, 0, 1)^T \in S$ but their sum $(1, 0, 1)^T \notin S$ as $1 + 1 = 2 \neq 1$.

(b) $S = \{\mathbf{x} = (x_1, x_2)^T : x_1 = x_2 = x_3\}$. (i) $\mathbf{0} \in S$ since $0 = 0 = 0$ so S is not empty. (ii) If $\mathbf{x}, \mathbf{y} \in S$, i.e, $x_1 = x_2 = x_3$, $y_1 = y_2 = y_3$, then $\mathbf{x} + \mathbf{y} \in S$ since $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)^T$ and $x_1 + y_1 = x_2 + y_2 = x_3 + y_3$. (iii) If $\mathbf{x} \in S$ and $\alpha \in \mathbb{R}$, i.e, $x_1 = x_2 = x_3$, then $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \alpha x_3) \in S$ since $\alpha x_1 = \alpha x_2 = \alpha x_3$.
So S is a subspace of $\mathbb{R}^3$.

(c) $S = \{\mathbf{x} = (x_1, x_2, x_3)^T : x_3 = x_1 + x_2\}$. (i) $\mathbf{0} \in S$ since $0 = 0 + 0$ so S is not empty. (ii) If $\mathbf{x}, \mathbf{y} \in S$, i.e, $x_3 = x_1 + x_2$, $y_3 = y_1 + y_2$, then $\mathbf{x} + \mathbf{y} \in S$ since $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, x_3 + y_3)^T$ and $x_3 + y_3 = (x_1 + x_2) + (y_1 + y_2) = (x_1 + y_1) + (x_2 + y_2)$. (iii) If $\mathbf{x} \in S$ and $\alpha \in \mathbb{R}$, i.e, $x_3 = x_1 + x_2$, then $\alpha\mathbf{x} = (\alpha x_1, \alpha x_2, \alpha x_3) \in S$ since $\alpha x_3 = \alpha(x_1 + x_2) = \alpha x_1 + \alpha x_2$.
So S is a subspace of $\mathbb{R}^2$.

3 (a) Let S be the set of all 2×2 diagonal matrices. (i) The zero matrix is in S. (ii) The sum of two diagonal matrices is still a diagonal matrix. (iii) The scalar multiple of a diagonal matrix is still a diagonal matrix.
So S is a subspace of $\mathbb{R}^{2\times 2}$.

(b) The set of 2×2 triangular matrices is a subspace of $\mathbb{R}^{2\times 2}$ and it is similar to (a).

(c) Similar to (b).

(d) Let S be the set of all 2×2 matrices with $a_{12} = 1$. (i) The zero matrix is not in S. So it is not a subspace of $\mathbb{R}^{2\times 2}$ (if it were, then it would contain the zero matrix (the identity element of $\mathbb{R}^{2\times 2}$)).

4 (a)

$$[A \mid \mathbf{0}] = \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 3 & 2 & 0 \end{array}\right] \quad R_2 - \tfrac{3}{2}R_1 \quad \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 0 \end{array}\right]$$

so the null space of A is $\{(0,0)^T\}$.

(b)

$$[A \mid \mathbf{0}] = \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ -2 & -4 & 6 & 3 & 0 \end{array}\right] \quad R_2 + 2R_1 \quad \left[\begin{array}{cccc|c} 1 & 2 & -3 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

so the null space of A is $\{(-2\alpha + 3\beta, \alpha, \beta, 0)^T : \alpha, \beta \in \mathbb{R}\}$.

9 (a) Set $\alpha(2,1)^T + \beta(3,2)^T = (a,b)^T \in \mathbb{R}^2$. Consider the augmented matrix

$$\left[\begin{array}{cc|c} 2 & 3 & a \\ 1 & 2 & b \end{array}\right] \quad R_2 - \tfrac{1}{2}R_1 \quad \left[\begin{array}{cc|c} 1 & 3 & a \\ 0 & \frac{1}{2} & b - \frac{1}{2}a \end{array}\right]$$

The system is consistent ($\beta = 2b - a$ and $\alpha = a - 3(2b - a) = 4a - 6b$). So the set is a spanning set.

(c) Set $\alpha(-2,1)^T + \beta(1,3)^T + \gamma(2,4)^T = (a,b)^T \in \mathbb{R}^2$. Consider the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ 1 & 3 & 4 & b \end{array}\right] \quad R_2 + \tfrac{1}{2}R_1 \quad \left[\begin{array}{ccc|c} -2 & 1 & 2 & 1 \\ 0 & \frac{5}{2} & 5 & b + \frac{1}{2} \end{array}\right]$$

The system is consistent so that the set is a spanning set.

10 (a) Consider the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 1 & 1 & c \end{array}\right] \quad R_3 - R_2 \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c - b \end{array}\right] \quad R_1 - R_3 \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & a - c + b \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c - b \end{array}\right].$$

In other words, the system is consistent so that it is a spanning set for $\mathbb{R}^3$. Indeed $(a,b,c)^T = (a - c + b)(1,0,0)^T + b(0,1,1)^T + (c - b)(1,0,1)^T$.

(b) Consider the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 2 & b \\ 0 & 1 & 1 & 3 & c \end{array}\right] \quad R_3 - R_2 \quad \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & a \\ 0 & 1 & 0 & 2 & b \\ 0 & 0 & 1 & 1 & c - b \end{array}\right] \quad R_1 - R_3 \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & a - c + b \\ 0 & 1 & 0 & 2 & b \\ 0 & 0 & 1 & 1 & c - b \end{array}\right].$$

It has infinitely many solution, but the point is that it is consistent. So it is a spanning set for $\mathbb{R}^3$.

(e) Consider the augmented matrix

$$\left[\begin{array}{cc|c} 1 & 0 & a \\ 1 & 2 & b \\ 3 & 1 & c \end{array}\right] \quad \begin{array}{c} R_2 - R_1 \\ R_3 - 3R_1 \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b - a \\ 0 & 1 & c - 3a \end{array}\right] \quad R_2 - \tfrac{1}{2}R_2 \quad \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 2 & b - a \\ 0 & 0 & c - 3a - \frac{1}{2}b + \frac{1}{2}a \end{array}\right].$$

It has no solution if $c - \frac{5}{2}a - \frac{1}{2}b \neq 0$, say $(a,b,c)^T = (0,0,1)^T$. So it is not a spanning set for $\mathbb{R}^3$.

11 (a) $\mathbf{x} \in \text{span } \{\mathbf{x_1}, \mathbf{x_2}\}$ means that $\mathbf{x} = \alpha\mathbf{x_1} + \beta\mathbf{x_2}$ for some $\alpha, \beta \in \mathbb{R}$. Consider the augmented matrix

$$[\mathbf{x_1}\ \mathbf{x_2}|\mathbf{x}] = \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 2 & 4 & 6 \\ 3 & 2 & 6 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + 3R_1 \end{array} \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 10 & 10 \\ 0 & 11 & 13 \end{array}\right] \begin{array}{l} \\ R_2/10 \\ \end{array}$$

$$\left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 11 & 13 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 11R_2 \end{array} \left[\begin{array}{cc|c} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{array}\right]$$

It is inconsistent. So $\mathbf{x} \notin \text{span } \{\mathbf{x_1}, \mathbf{x_2}\}$.

(b) $\mathbf{y} \in \text{span } \{\mathbf{x_1}, \mathbf{x_2}\}$ means that $\mathbf{y} = \alpha\mathbf{x_1} + \beta\mathbf{x_2}$ for some $\alpha, \beta \in \mathbb{R}$. Consider the augmented matrix

$$[\mathbf{x_1}\ \mathbf{x_2}|\mathbf{y}] = \left[\begin{array}{cc|c} -1 & 3 & -9 \\ 2 & 4 & -2 \\ 3 & 2 & 5 \end{array}\right] \begin{array}{l} \\ R_2 + 2R_1 \\ R_3 + 3R_1 \end{array} \left[\begin{array}{cc|c} -1 & 3 & -9 \\ 0 & 10 & -20 \\ 0 & 11 & -22 \end{array}\right] \begin{array}{l} \\ R_2/10 \\ \end{array}$$

$$\left[\begin{array}{cc|c} -1 & 3 & -9 \\ 0 & 1 & -2 \\ 0 & 11 & -22 \end{array}\right] \begin{array}{l} \\ \\ R_3 - 11R_2 \end{array} \left[\begin{array}{cc|c} -1 & 3 & -9 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{array}\right]$$

It is consistent and $\alpha = 3$ and $\beta = -2$. So $\mathbf{y} \in \text{span } \{\mathbf{x_1}, \mathbf{x_2}\}$.

12 Let $\{\mathbf{x_1}, \ldots, \mathbf{x_n}\}$ be a spanning set for a vector space V.

(a) If we add an additional vector $\mathbf{x}_{k+1}$ to the spanning set, we still have a spanning set. It it because each $\mathbf{x} \in V$ can be written as a linear combination of $\mathbf{x_1}, \ldots, \mathbf{x_k}$, say, $\mathbf{x} = \alpha_1\mathbf{x_1} + \cdots + \alpha_k\mathbf{x_k}$ so that we have $\mathbf{x} = \alpha_1\mathbf{x_1} + \cdots + \alpha_k\mathbf{x_k} + 0 \cdot \mathbf{x_{k+1}}$.

(b) In general it is not true, for example $\mathbf{e}_1, \mathbf{e}_2$ span $\mathbb{R}^2$ but $\mathbf{e}_1$ does not span $\mathbb{R}^2$. However it can be true for **some** situations, for example $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2$ span $\mathbb{R}^2$ (by (a)) and if we remove $\mathbf{e}_1 + \mathbf{e}_2$, we still have a spanning set.