Math 2660 Topics in Linear Algebra, Key

1.5

1,4a,c,6,8,9,11,13,15

1 (a) $A^{-1}[A \quad I] = [A^{-1}A \quad A^{-1}I] = [I \quad A^{-1}]$.

(b) $\begin{bmatrix} A \\ I \end{bmatrix} A^{-1} = \begin{bmatrix} AA^{-1} \\ A^{-1} \end{bmatrix} = \begin{bmatrix} I \\ A^{-1} \end{bmatrix}$.

(c) $[A \quad I]^T[A \quad I] = \begin{bmatrix} A^T \\ I \end{bmatrix} [A \quad I] = \begin{bmatrix} A^TA & A^T \\ A & I \end{bmatrix}$.

(d) $[A \quad I][A \quad I]^T = [A \quad I] \begin{bmatrix} A^T \\ I \end{bmatrix} = AA^T + I$.

(e) $\begin{bmatrix} A^{-1} \\ I \end{bmatrix} [A \quad I] = \begin{bmatrix} A^{-1}A & A^{-1} \\ A & I \end{bmatrix} = \begin{bmatrix} I & A^{-1} \\ A & I \end{bmatrix}$.

4 (a) $\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{21} & B_{22} \\ B_{11} & B_{12} \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$.

(c) $\begin{bmatrix} D & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} DB_{11} & DB_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 3 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix}$ since $D = 2I$.

6 (a) $XY^T = x_1y_1^T + x_2y_2^T + x_3y_3^T = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \end{bmatrix} + \begin{bmatrix} 5 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 20 & 5 \\ 12 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 20 & 17 \end{bmatrix}$.

(b) $YX^T = y_1x_1^T + y_2x_2^T + y_3x_3^T = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} + \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 20 & 12 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 24 & 12 \\ 20 & 17 \end{bmatrix}$.

YX^T is the transpose of XY^T. Indeed we can know prior to the above detailed computation since $(XY^T)^T = (Y^T)^TX^T = YX^T$.

8 Let $X = [x_1 \cdots x_r]$ and $B = [b_1 \cdots b_r]$ be the column forms of X and B respectively. Then

$$AX = B \Leftrightarrow A[x_1 \cdots x_r] = [b_1 \cdots b_r] \Leftrightarrow [Ax_1 \cdots Ax_r] = [b_1 \cdots b_r]$$

i.e., $Ax_i = b_i$ for all $i = 1, \dots, r$.

9 (a) $D = \begin{bmatrix} d_{11} & & \\ & \ddots & \\ & & d_{nn} \end{bmatrix} = [d_{11}e_1 \cdots d_{nn}e_n]$.

(b) Let $A = [a_1 \cdots a_n]$ be in column form. Then

$$AD = A[d_{11}e_1 \cdots d_{nn}e_n] = [d_{11}Ae_1 \cdots d_{nn}Ae_n] = [d_{11}a_1 \cdots d_{nn}a_n],$$

since $Ae_i = a_i$, $i = 1, \ldots, n$. In other words the post multiplication of D on A (i.e., multiply A by D form the right), where D is a diagonal matrix, means multiplying the columns of A by the corresponding diagonal entries of D.

11 (a) Consider the system of linear equations $\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then we have

$$A_{11}x_1 + A_{12}x_2 = 0 \quad (1)$$

$$A_{22}x_2 = 0 \quad (2)$$

From (2) $x_2 = 0$ since A_{22} is nonsingular. Then from (1), $A_{11}x_1 = 0$ so that $x_1 = 0$ since A_{11} is nonsingular. So the system has only trivial solution. Thus by Theorem 1.4.2, A is nonsingular.

Let $A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$. Then $A^{-1}A = I$ amounts to

$$\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

So

$$\begin{bmatrix} B_{11}A_{11} & B_{11}A_{12} + B_{12}A_{22} \\ B_{21}A_{11} & B_{21}A_{12} + B_{22}A_{22} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

Hence we have

$$B_{11}A_{11} = I \quad (3)$$

$$B_{11}A_{12} + B_{12}A_{22} = 0 \quad (4)$$

$$B_{21}A_{11} = 0 \quad (5)$$

$$B_{21}A_{12} + B_{22}A_{22} = I \quad (6)$$

From (3) $B_{11} = A_{11}^{-1}$. From (5) $B_{21} = 0 \cdot A_{11}^{-1} = 0$. From (6) $B_{22} = A_{22}^{-1}$. From (4) $B_{12} = -(B_{11}A_{12})A_{22}^{-1} = -A_{11}^{-1}A_{12}A_{22}^{-1}$. Hence $A^{-1} = \begin{bmatrix} A_{11}^{-1} & -A_{11}^{-1}A_{12}A_{22}^{-1} \\ 0 & A_{22}^{-1} \end{bmatrix}$.

(b) So $C = -A_{11}^{-1}A_{12}A_{22}^{-1}$.

13 $A = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix}$.

$A^4 = A^2A^2 = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} B^2 & 0 \\ 0 & B^2 \end{bmatrix}$. In general $A^{2n} = \begin{bmatrix} B^n & 0 \\ 0 & B^n \end{bmatrix}$ where n is a positive integer.

15 $S = \begin{bmatrix} I & A \\ 0 & I \end{bmatrix}$. By Exercise 11, $S^{-1} = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix}$. Then

$$S^{-1}MS = \begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I & A \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ B & AB \end{bmatrix} = M'(\text{say}).$$

Later when we study eigenvalues and similarity we can say M and M' are similar and thus AB and BA have the same eigenvalues. But we don't need to worry about it, at least now.