Math 2660 Topics in Linear Algebra, Key

1.4

1, 3, 5-7, 8a,c, 9,10a,g,11a, 12a

1 (a) type I, (c) type III, (d) type II are elementary matrices but (b) is not and it is a product of two type II elementary matrices.

3 (a) $E = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$, (b) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, (c) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & 1 & 0 \end{bmatrix}$.

5 (a) $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

(b) $F = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$.

(c) C is row equivalent to A since $FEA = C$, i.e., two elementary row operations on A yield C.

6 (a) $\begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \begin{matrix} \\ \\ R_3 + R_2 \end{matrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} = U$. So

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

(b) $E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

So $L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$ is lower triangular. $A = LU$ by direct verification.

7 $\begin{bmatrix} 2 & 1 \\ 6 & 4 \end{bmatrix} \begin{matrix} \\ R_2 - 3R_1 \end{matrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{matrix} \frac{1}{2}R_1 \\ \\ \end{matrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{matrix} R_1 - \frac{1}{2}R_2 \\ \\ \end{matrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. So

$$E_1 = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}.$$

(a) $A = E_1^{-1}E_2^{-1}E_3^{-1} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$.

(b) $A^{-1} = E_3E_2E_1 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$.

8 (a) $\begin{bmatrix}3&1\\9&5\end{bmatrix}$ R_2-3R_1 $\begin{bmatrix}3&1\\0&2\end{bmatrix}=U$. So $L=E_1^{-1}=\begin{bmatrix}1&0\\3&1\end{bmatrix}$.

(b) $\begin{bmatrix}1&1&1\\3&5&6\\-2&2&7\end{bmatrix}$ $\begin{matrix}R_2-3R_1\\R_3+2R_1\end{matrix}$ $\begin{bmatrix}1&1&1\\0&2&3\\0&4&9\end{bmatrix}$ R_3-2R_2 $\begin{bmatrix}1&1&1\\0&2&3\\0&0&3\end{bmatrix}=U$.

So $L=E_1^{-1}E_2^{-1}E_3^{-1}=\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-2&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-3&1&0\\2&-2&1\end{bmatrix}$.

9 Let $A=\begin{bmatrix}1&0&1\\3&3&4\\2&2&3\end{bmatrix}$.

(a) To verify, just check $AA^{-1}=I$ or $A^{-1}A=I$. To find:

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\3&3&4&0&1&0\\2&2&3&0&0&1\end{array}\right]\ \begin{matrix}R_2-3R_1\\R_3-2R_1\end{matrix}\ \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&3&1&-3&1&0\\0&2&1&-2&0&1\end{array}\right]\ \tfrac{1}{3}R_2$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{1}{3}&-1&\frac{1}{3}&0\\0&2&1&-2&0&1\end{array}\right]\ R_3-2R_2\ \left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{1}{3}&-1&\frac{1}{3}&0\\0&0&\frac{1}{3}&0&-\frac{2}{3}&1\end{array}\right]\ 3R_3$$

$$\left[\begin{array}{ccc|ccc}1&0&1&1&0&0\\0&1&\frac{1}{3}&-1&\frac{1}{3}&0\\0&0&1&0&-2&3\end{array}\right]\ \begin{matrix}R_1-R_3\\R_2-\frac{1}{3}R_3\end{matrix}\ \left[\begin{array}{ccc|ccc}1&0&0&1&2&-3\\0&1&0&-1&1&-1\\0&0&1&0&-2&3\end{array}\right]$$

So $A^{-1}=\begin{bmatrix}1&2&-3\\-1&1&-1\\0&-2&3\end{bmatrix}$.

(b) (i) $x=A^{-1}b=\begin{bmatrix}1&2&-3\\-1&1&-1\\0&-2&3\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\-1\\1\end{bmatrix}$. (ii) and (iii) are similar.

10 (a)

$$\left[\begin{array}{cc|cc}-1&1&1&0\\1&0&0&1\end{array}\right]\ -R_1\ \left[\begin{array}{cc|cc}1&-1&-1&0\\1&0&0&1\end{array}\right]\ R_2-R_1\ \left[\begin{array}{cc|cc}1&-1&-1&0\\0&1&1&1\end{array}\right]$$

$$R_1+R_2\ \left[\begin{array}{cc|cc}1&0&0&1\\0&1&1&1\end{array}\right]$$

so that $A^{-1}=\begin{bmatrix}0&1\\1&1\end{bmatrix}$. (g) is similar.

11 (a) $A^{-1}=\begin{bmatrix}2&-1\\-5&3\end{bmatrix}$ so $X=A^{-1}B=\begin{bmatrix}2&-1\\-5&3\end{bmatrix}\begin{bmatrix}1&2\\3&4\end{bmatrix}=\begin{bmatrix}-1&0\\4&2\end{bmatrix}$.

12 (a) $A^{-1}=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}$ so $X=A^{-1}(C-B)=\begin{bmatrix}2&-3\\-3&5\end{bmatrix}\begin{bmatrix}-2&-4\\-8&-1\end{bmatrix}=\begin{bmatrix}20&-5\\-34&7\end{bmatrix}$.