Math 2660 Topics in Linear Algebra, Key

1.3

1-8,10-15

1 (a) $2A = 2\begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 8 \\ -4 & 0 & 2 \\ 2 & 4 & 4 \end{bmatrix}$.

(b) $A + B = \begin{bmatrix} 3 & 1 & 4 \\ -2 & 0 & 1 \\ 1 & 2 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 1 \\ 2 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 & 6 \\ -5 & 1 & 2 \\ 3 & -2 & 3 \end{bmatrix}$.

Others are similar

2 (a) $\begin{bmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3 \cdot 2 + 5 \cdot 1 + 1 \cdot 4 & 3 \cdot 1 + 5 \cdot 3 + 1 \cdot 1 \\ (-2) \cdot 2 + 0 \cdot 1 + 2 \cdot 4 & (-2) \cdot 1 + 0 \cdot 2 + 2 \cdot 1 \end{bmatrix} = \begin{bmatrix} 15 & 19 \\ 4 & 0 \end{bmatrix}$.

Others are similar.

3 (a) 3×3.

(b) 1×2.

Others are undefined.

4. (a) $\begin{bmatrix} 3 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$.

(b) $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & -1 \\ 3 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix}$.

(c) $\begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}$.

5-8 Direct verifications.

10 $A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Hence $A^2 = \frac{1}{4}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = A$. Hence $A^3 = A$ and in general $A^n = A$ for all positive integers n.

11 $A = \frac{1}{2}\begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$. Hence $A^2 = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = I$. So $A^3 = A$ and in general $A^{2n} = I$ and $A^{2n+1} = A$ for all positive integers n.

12 Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Then $A^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^4 = 0$.

13 (a) $b = 2a_1 + a_2$ by inspection.

(b) Write $b = x_1a_1 + x_2a_2$ means $Ax = b$. So $x = (2, 1)$.

(c) Solving

$$\left[\begin{array}{cc|c} 1 & 2 & -3 \\ 1 & -2 & -2 \end{array}\right] \quad R_2 - R_1 \quad \left[\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & -4 & 1 \end{array}\right]$$

so that $x = (-\frac{5}{2}, -\frac{1}{4})$ by back substitution. Hence $c = -\frac{5}{2}a_1 - \frac{1}{4}a_2$.

14 (a) The two columns of A are scalar multiple to each other. But b is not a scalar multiple of either. So inconsistent.

(b) $b = a_1 + a_2$. So consistent.

(c) The three columns of A are scalar multiple to each other. But b is not a scalar multiple of any. So inconsistent.

15

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{d} \begin{bmatrix} a_{11}a_{22} - a_{21}a_{12} & 0 \\ 0 & a_{11}a_{22} - a_{21}a_{12} \end{bmatrix} = I.$$

Similarly

$$\frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = I.$$

So $A^{-1} = \frac{1}{d} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$.