Math 2660 Topics in Linear Algebra, Key

1.2

1-4,5a,c,e,g,i,k,7,8,9,10

1 (a), (c), (d), (g), (h) are in row echelon form but the others are not.

2 (a), (e) are inconsistent. The others are consistent. (b) and (d) have unique solution.

(b) By back substitution, $x = (4, -1)$.

(d) By back substitution, $x = (4, 5, 2)$.

3 (a) Read the solution off from the matrix, $x = (3, 5, -2)$.

(b) inconsistent system.

(c) $x_3 = -2$, $x_2 = \alpha$, $x_1 = 2 + 3\alpha$. Thus $x = (2 + 3\alpha, \alpha, -2)$.

(d) $x_4 = \beta$, $x_3 = 4 - 3\beta$, $x_2 = \alpha$, $x_1 = 5 - 2\alpha - \beta$. Thus $x = (5 - 2\alpha - \beta, \alpha, 4 - 3\beta, \beta)$.

(e) $x_4 = 6$, $x_3 = \beta$, $x_2 = \alpha$, $x_1 = 3 + 2\beta - 5\alpha$. Thus $x = (3 + 2\beta - 5\alpha, \alpha, \beta, 6)$.

(f) $x_3 = -1$, $x_2 = 2$ and $x_1 = \alpha$. Thus $x = (\alpha, 2, -1)$.

4. (a) x_1, x_2, x_3 are lead variables.

(b) x_1 and x_3 are lead variables and x_2 is a free variable.

(c) x_1 and x_3 are lead variables and x_2 is a free variable.

(d) x_1 and x_3 are lead variables and x_2, x_4 are free variables.

(e) x_1 and x_4 are lead variables and x_2 and x_3 are free variables.

(f) x_2 and x_3 are lead variables and x_1 is a free variable.

5 (a)

$$\left[\begin{array}{cc|c} 1 & -2 & 3 \\ 2 & -1 & 9 \end{array}\right] \; R_2 - 2R_1 \; \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 3 & 3 \end{array}\right] \; \tfrac{1}{3}R_2 \; \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 1 \end{array}\right]$$

So $x = (1, 5)$.

(c)

$$\left[\begin{array}{cc|c} 1 & 1 & 0 \\ 2 & 3 & 0 \\ 3 & -2 & 0 \end{array}\right] \; \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \; \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & -5 & 0 \end{array}\right] \; R_3 + 5R_1 \; \left[\begin{array}{cc|c} 1 & -2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

So $x = (0, 0, 0)$.

(e)

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 1 \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \; \tfrac{1}{2}R_1 \; \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 1 & 1 & 3 \\ 3 & 4 & 2 & 4 \end{array}\right] \; \begin{array}{c} \\ R_2 - R_1 \\ R_3 - 3R_1 \end{array} \; \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \end{array}\right] \; R_3 - R_2$$

$$\left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right] \; -2R_2 \; \left[\begin{array}{ccc|c} 1 & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So $x = (8 - 2\alpha, \alpha - 5, \alpha)$ by back substitution.

(g)

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 2 & 3 & -1 & -1 & 2 \\ 3 & 2 & 1 & 1 & 5 \\ 3 & 6 & -1 & -1 & 4 \end{array}\right] \begin{array}{c} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 3R_1 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 2 \\ 0 & -1 & -2 & -2 & 5 \\ 0 & 3 & -4 & -4 & 4 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \\ R_4 - 3R_2 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 2 \\ 0 & 0 & -5 & -5 & 7 \\ 0 & 0 & 5 & 5 & -2 \end{array}\right] \begin{array}{c} \\ \\ \\ R_4 + R_3 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 2 \\ 0 & 0 & -5 & -5 & 7 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right] \begin{array}{c} \\ \\ -\frac{1}{5}R_3 \\ \frac{1}{5}R_4 \end{array} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & -3 & -3 & 2 \\ 0 & 0 & 1 & 1 & -\frac{7}{5} \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

Inconsistent.
(i) and (k) are similar.

7 A homogenous 2×3 system represents two planes passing through the origin. So the intersection is either a line or a plane and thus the system has infinitely many solutions.
A nonhomogeneous 2×3 system represents two planes; one of them (or both) does not pass through the origin. Either there is

1. no solution (parallel planes)
2. infinitely many solutions (they intersect, i.e., either a line or a plane).

However it is impossible to have exactly one solution. One can see that also from the row echelon form of the augmented matrix.

8

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ -1 & 4 & 3 & 2 \\ 2 & -2 & a & 3 \end{array}\right] \begin{array}{c} \\ R_2 + R_1 \\ R_3 - 2R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & -6 & a-2 & 1 \end{array}\right] \begin{array}{c} \\ \\ R_3 + R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 1 \\ 0 & 6 & 4 & 3 \\ 0 & 0 & a+2 & 4 \end{array}\right]$$

In order to have a unique solution, $a + 2 \neq 0$, i.e., $a \neq -2$.

9 (a) Always consistent because it is homogenous and thus has the trivial solution.
(b)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & 0 \\ -1 & 1 & \beta & 0 \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 + R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 3 & \beta+1 & 0 \end{array}\right] \begin{array}{c} \\ \frac{1}{4}R_2 \\ \\ \end{array} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 3 & \beta+1 & 0 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 3R_2 \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{4} & 0 \\ 0 & 0 & \beta+\frac{1}{4} & 0 \end{array}\right]$$

In order to have infinitely many solutions, $\beta \neq -\frac{1}{4}$.

10

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & a & b \end{array}\right] \begin{array}{c} \\ R_2 - R_1 \\ R_3 - R_1 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & a-3 & b-2 \end{array}\right] \begin{array}{c} \\ \\ R_3 - 2R_2 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-5 & b-3 \end{array}\right]$$

(a) In order to have infinitely many solutions, $a - 5 = b - 3 = 0$, i.e., $a = 5$ and $b = 3$.
(b) In order to have inconsistent system, $a = 5$ and $b \neq 3$.