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Heinz-Kato's inequalities for semisimple Lie groups

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Abstract. Extensions of Heinz-Kato's inequalities and related inequalities are obtained for semisimple connected noncompact Lie groups.

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1. Introduction

Let $\mathbb{C}_{n\times n}$ be the set of $n \times n$ complex matrices and let

$$\|X\| := \max_{\|v\|_2=1} \|Xv\|_2$$

denote the spectral norm of $X \in \mathbb{C}_{n\times n}$. We have the following norm inequalities.

Theorem 1.1. 1. (Heinz-Kato [16, Theorem 3]) If $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite and $X \in \mathbb{C}_{n\times n}$, then

$$\|A^t X B^t\| \leq \|X\|^{1-t} \|AXB\|^t, \qquad 0 \leq t \leq 1. \tag{1}$$

2. (Heinz-Kato [17]) If $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite and $X \in \mathbb{C}_{n\times n}$, then

$$\|A^t X B^{1-t}\| \leq \|AX\|^t \|XB\|^{1-t}, \qquad 0 \leq t \leq 1, \tag{2}$$

3. (McIntosh [23], Bhatia-Davis [4]) For $A, B, X \in \mathbb{C}_{n\times n}$,

$$\|A^* X B\| \leq \|AA^* X\|^{1/2} \|XBB^*\|^{1/2}. \tag{3}$$

See [1, 9, 10, 15, 16, 17, 18, 25] for the inequalities and related inequalities.

By continuity it suffices to consider the general linear group $\mathrm{GL}_n(\mathbb{C})$ instead of $\mathbb{C}_{n\times n}$. Moreover by the homogeneous property of $\|\cdot\|$ we can restrict ourselves to $\mathrm{SL}_n(\mathbb{C})$.

We will obtain some extensions of the above inequalities and other related inequalities in the context of semisimple connected noncompact Lie groups.

2. Log majorization

If we order the singular values of $X \in \mathbb{C}_{n\times n}$ in descending order

$$s_1(X) \geq \cdots \geq s_n(X),$$

then (2), for example, can be written as

$$s_1(A^t X B^{1-t}) \leq s_1^t(AX) s_1^{1-t}(XB), \qquad 0 \leq t \leq 1. \tag{4}$$

Let $\mathbb{R}_+^n$ denote the set of positive n-tuples and let $a, b \in \mathbb{R}_+^n$. Then a is said to be *log majorized* by b, denoted by $a \prec_{\log} b$ if

$$\begin{aligned} \max_{\sigma\in S_n} \prod_{i=1}^k a_{\sigma(i)} &\leq \max_{\sigma\in S_n} \prod_{i=1}^k b_{\sigma(i)}, \quad k = 1, \ldots, n-1, \\ \prod_{i=1}^n a_i &= \prod_{i=1}^n b_i, \end{aligned}$$

where S_n denotes the symmetric group on $\{1, \ldots, n\}$. Write

$$s(X) := (s_1(X), \ldots, s_n(X)), \quad s^t(X) := (s_1^t(X), \ldots, s_n^t(X)), \quad t \geq 0.$$

Suppose $1 \leq k \leq n$. The kth compound of $X \in \mathbb{C}_{n\times n}$ is defined to be the $\binom{n}{k} \times \binom{n}{k}$ complex matrix $C_k(X)$ [22] whose elements are defined by

$$C_k(X)_{\alpha,\beta} = \det X[\alpha|\beta],$$

where $\alpha, \beta \in Q_{k,n}$ and $Q_{k,n} = \{\omega = (\omega(1), \ldots, \omega(k)) : 1 \leq \omega(1) < \cdots < \omega(k) \leq n\}$ is the set of increasing sequences of length k chosen from $1, \ldots, n$. For example, if $n = 3$ and $k = 2$, then

$$C_2(X) = \begin{pmatrix} \det X[1,2|1,2] & \det X[1,2|1,3] & \det X[1,2|2,3] \\ \det X[1,3|1,2] & \det X[1,3|1,3] & \det X[1,3|2,3] \\ \det X[2,3|1,2] & \det X[2,3|1,3] & \det X[2,3|2,3] \end{pmatrix}.$$

In general $C_1(X) = X$ and $C_n(X) = \det X$. Compound matrix has very nice properties: (i) $C_k(AB) = C_k(A)C_k(B)$ for all $A, B \in \mathbb{C}_{n\times n}$, (ii) the eigenvalues of $C_k(X)$ are $\prod_{j=1}^k \lambda_{\omega(j)}(X)$, $\omega \in Q_{k,n}$, where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of X, (iii) the singular values of $C_k(X)$ are $\prod_{j=1}^k s_{\omega(j)}(X)$, $\omega \in Q_{k,n}$. The compound matrix $C_k(X)$ is indeed the matrix representation (with respect to some induced basis) of the induced operator $C_k(T)$ on the exterior space $\wedge^k \mathbb{C}^n$ where $T : \mathbb{C}^n \to \mathbb{C}^n$ is an operator: $C_k(T) v_1 \wedge \cdots \wedge v_k = T v_1 \wedge \cdots \wedge T v_k$, $v_1, \ldots, v_k \in \mathbb{C}^n$.

The following is an extension of Theorem 1.1. We will prove the second inequality and the rest are similar.

Theorem 2.1. 1. If $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite and $X \in \mathbb{C}_{n\times n}$ and $0 \leq t \leq 1$, then

$$s(A^t X B^t) \prec_{\log} s^{1-t}(X) s^t(AXB).$$

2. If $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite and $X \in \mathbb{C}_{n\times n}$ and $0 \le t \le 1$, then
$$s(A^tXB^{1-t}) \prec_{\log} s^t(AX)s^{1-t}(XB).$$
3. For $A, B, X \in \mathbb{C}_{n\times n}$,
$$s(A^*XB) \prec_{\log} s^{1/2}(AA^*X)s^{1/2}(XBB^*).$$

Proof. Let $C_k(X) \in \mathbb{C}_{\binom{n}{k}\times\binom{n}{k}}$ denote the kth compound of X, $k = 1, \ldots, n$. Notice that $s_1(C_k(X)) = \prod_{i=1}^k s_i(X)$, $C_k(XY) = C_k(X)C_k(Y)$, $X, Y \in \mathbb{C}_{n\times n}$, and $C_k(X)$ is positive semi-definite if X is positive semi-definite. So

$$\begin{array}{rcll} \prod_{i=1}^k s_i(A^tXB^{1-t}) & = & s_1(C_k(A^tXB^{1-t})) & \\ & = & s_1(C_k(A)^tC_k(X)C_k(B)^{1-t}) & \\ & \le & s_1^t(C_k(AX))s_1^{1-t}(C_k(XB)) & \text{by (4)} \\ & = & \prod_{i=1}^k s_i^t(AX)\prod_{i=1}^k s_i^{1-t}(XB). & \end{array}$$

When $k = n$,
$$\prod_{i=1}^n s_i(A^tXB^{1-t}) = |\det(A^tXB^{1-t})| = (\det A)^t\,|\det X|\,(\det B)^{1-t}$$
and
$$\begin{array}{rcl} \prod_{i=1}^n s_i^t(AX)\prod_{i=1}^n s_i^{1-t}(XB) & = & |\det(AX)|^t\,|\det(XB)|^{1-t} \\ & = & (\det A)^t\,|\det X|\,(\det B)^{1-t}. \end{array}$$

■

3. A pre-order of Kostant

We now take a close look of Theorem 2.1 for $X \in \mathrm{GL}_n(\mathbb{C})$. Let $A_+ \subset \mathrm{GL}_n(\mathbb{C})$ denote the set of all positive diagonal matrices with diagonal entries in descending order. Recall that the singular value decomposition of $X \in \mathrm{GL}_n(\mathbb{C})$ asserts that there exist unitary matrices U, V such that
$$X = Ua_+(X)V, \tag{5}$$
where $a_+(X) = \mathrm{diag}\,(s_1(X), \ldots, s_n(X)) \in A_+$. Though U and V in the decomposition (5) are not unique, $a_+(X) \in A_+$ is uniquely defined.

Let G be a semisimple noncompact connected Lie group having $\mathfrak{g}$ as its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the analytic subgroup with Lie algebra $\mathfrak{k}$. Then $\mathrm{Ad}\,K$ is a maximal compact subgroup of $\mathrm{Ad}\,G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace. The exponential map $\exp : \mathfrak{a} \to A$ is bijective.

Set

$$P := \exp \mathfrak{p}.$$

The map $K \times P \to G$, $(k,p) \mapsto kp$ is a diffeomorpism. In particular

$$G = KP$$

and every element $g \in G$ can be uniquely written as

$$g = kp, \qquad k \in K,\ p \in P. \tag{6}$$

The map $\Theta : G \to G$

$$\Theta(kp) = kp^{-1}, \quad k \in K,\ p \in P,$$

is an automorphism of G [19, p.387]. The map $* : G \to G$ defined by

$$g^* := \Theta(g^{-1}) = pk^{-1}, \qquad g \in G$$

is clearly a diffeomorphism. Let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. The Weyl group operates naturally in $\mathfrak{a}$ and A and the isomorphism $\exp : \mathfrak{a} \to A$ is a W-isomorphism.

Fix a *closed* Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and set $A_+ := \exp \mathfrak{a}_+$. We have [11] the Cartan decomposition

$$G = KA_+K.$$

Though $k_1, k_2 \in K$ are not unique in $g = k_1 a k_2$ ($g \in G$, $k_1, k_2 \in K$, $a \in A_+$), the element $a = a_+(g) \in A_+$ is unique.

Proposition 3.1. The following maps are continuous.

1. $a'_+ : \mathfrak{p} \to \mathfrak{a}_+$ where for each $X \in \mathfrak{p}$, $a'_+(X)$ is the unique element in $\mathfrak{a}_+$ such that $a'_+(X) = \mathrm{Ad}(s)X$ for some $s \in K$. Indeed it is a contraction.

2. $a_+ : G \to A_+$ where $a_+(g) \in A_+$ is the unique element in $g = k_1 a_+(g) k_2 \in G$, where $k_1, k_2 \in K$.

Proof. (1) By Berezin-Gelfand's result [2], $a'_+(X+Y) \in a'_+(X) + \mathrm{conv}\, W a'_+(Y)$ for any $X, Y \in \mathfrak{p}$. So $a'_+(X) = a'_+(Y + (X-Y)) \in a'_+(Y) + \mathrm{conv}\, W(a'_+(X-Y))$. Hence $a'_+(X) - a'_+(Y) \in \mathrm{conv}\, W a'_+(X-Y)$. Also see [13, Corollary 3.10]. Let $\|\cdot\|$ be the norm on $\mathfrak{p}$ induced by the Killing form of $\mathfrak{g}$. Since $\|\cdot\|$ is K-invariant and strictly convex, $\|a'_+(X) - a'_+(Y)\| \le \|a'_+(X-Y)\| = \|X - Y\|$. So the map a'_+ is a contraction and thus continuous.

(2) Since the map $G \to P$ such that $g = kp \mapsto p$ is differentiable and $a_+(g) = a_+(p)$, it suffices to establish the continuity of $a_+ : P \to A_+$. The map $\exp : \mathfrak{p} \to P$ is a surjective diffeomorphism and the inverse $\log : P \to \mathfrak{p}$ is well defined. So $a_+ = \exp \circ a'_+ \circ \log$ on P is continuous. ■

Define a pre-order $\prec$ in A. Given $a, b \in A$, $a \prec b$ means

$$\exp(\mathrm{conv}\, W(\log a)) \subset \exp(\mathrm{conv}\, W(\log b)).$$

The set $\exp(\mathrm{conv}\, W(\log a))$ is multiplicatively the convex hull of the compact convex set having the Weyl group orbit $W(\log a)$ as its extreme points.

Example 3.2. Let $G = \mathrm{SL}(n, \mathbb{C})$. We pick

$$\begin{aligned}
\mathfrak{k} &= \mathfrak{su}(n),\\
K &= \mathrm{SU}(n),\\
\mathfrak{p} &= i\mathfrak{su}(n), \text{ i.e., the set of Hermitian matrices of zero trace}\\
P &= \text{the set of positive definite matrices in } \mathrm{SL}_n(\mathbb{C})\\
A &= \{\mathrm{diag}\,(a_1, \ldots, a_n) : a_1, \ldots, a_n > 0,\ \prod_{i=1}^n a_i = 1\},\\
A_+ &= \{\mathrm{diag}\,(a_1, \ldots, a_n) : a_1 \ge \cdots \ge a_n > 0,\ \prod_{i=1}^n a_i = 1\}.
\end{aligned}$$

Let $a = \mathrm{diag}\,(a_1, \ldots, a_n), b = \mathrm{diag}\,(b_1, \ldots, b_n) \in A$. Since the Weyl group is the symmetric group S_n on $\{1, \ldots, n\}$, $\mathrm{conv}\, W(\log a) = \mathrm{conv}\, S_n(\log a)$. So $a \prec b$ amounts to $\log a \in \mathrm{conv}\, S_n(\log b)$ and by Hardy-Littlewood-Poyla's theorem, $a \prec b$ is equivalent to the log majorization inequalities

$$\begin{aligned}
\prod_{i=1}^k a_{[i]} &\le \prod_{i=1}^k b_{[i]}, \quad k = 1, \ldots, n-1,\\
\prod_{i=1}^n a_{[i]} &= \prod_{i=1}^n b_{[i]},
\end{aligned}$$

where $a_{[1]} \ge \cdots \ge a_{[n]}$ denote the rearranged $a_1, \ldots, a_n$ in descending order.

The following nice result of Kostant describes the pre-order $\prec$ in A via the representations of G. We remark that Kostant's pre-order [21, p.426] is more general and is defined in G via the complete multiplicative Jordan decomposition and hyperbolic elements (see Section 5 and [21]).

Theorem 3.3. (Kostant [21, Theorem 3.1]) Let $f, g \in A$. Then $f \prec g$ if and only if $|\pi(f)| \le |\pi(g)|$ for all finite dimensional representations π of G, where $|\cdot|$ denotes the spectral radius.

One may derive the log majorization in Example 3.2 via Theorem 3.3 and the fundamental representations on the exterior space $\wedge^k \mathbb{C}^n$, $k = 1, \ldots, n-1$, since $|\pi_k(a)| = a_1 \cdots a_k$.

4. Extension of the inequalities

Lemma 4.1. Let $g \in G$. Then $a_+(g) = a_+(g^*) = a_+^{1/2}(gg^*) = a_+^{1/2}(g^*g)$.

Proof. Let $g = kp$ be the Cartan decomposition of $g \in G$, $k \in K$, $p \in P$. Notice that $g^* = pk^{-1}$ so that $a_+(g) = a_+(p) = a_+(g^*)$. Now $g^*g = p^2$ and $gg^* = kp^2k^{-1}$. So $a_+^{1/2}(gg^*) = a_+^{1/2}(g^*g) = a_+^{1/2}(p^2)$. Since each element in P is K-conjugate to some element in A_+ [19, p.320]. Thus $a_+^{1/2}(p^2) = a_+(p) = a_+(g)$. ■

Lemma 4.2. Let $h_1, h_2 \in A_+$. For any finite dimensional representation $\pi : G \to \mathrm{GL}(V)$, $|\pi(h_1 h_2)| = |\pi(h_1)|\,|\pi(h_2)|$, where $|\cdot|$ denotes the spectral radius.

Proof. Since the spectral radius of an operator is invariant under similarity, by the complete reducibility [5, p.50], [14, p.28] of π, we may assume that π is irreducible. We also use the same notation $d\pi$ to denote the irreducible representation of the complexification $\mathfrak{g}_{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$ (direct sum) of $\mathfrak{g}$, induced by $d\pi : \mathfrak{g} \to \mathfrak{gl}(V)$, i.e., $d\pi(X + iY) = d\pi(X) + i\, d\pi(Y)$, $X, Y \in \mathfrak{g}$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. Since $\mathfrak{u} := \mathfrak{k} + i\mathfrak{p}$ is a compact real form of $\mathfrak{g}_{\mathbb{C}}$, there is an inner product (unique up to scalar multiple) on V [6, p.217] such that $d\pi(\mathfrak{u})$ are skew Hermitian. So $d\pi(\mathfrak{k})$ are skew Hermitian and $d\pi(\mathfrak{p})$ are Hermitian. Thus the elements of

$$\pi(P) = \pi(\exp \mathfrak{p}) = \exp d\pi(\mathfrak{p})$$

[11, p.110] are positive definite operators. Since $A \subset P$ and is abelian, $\pi(A)$ is an abelian subgroup of positive definite operators. Thus the elements of $\pi(A)$ are positive diagonal operators under an appropriate orthonormal basis (once fixed and for all) of V. For each $H \in \mathfrak{a}$, $\exp d\pi(H) = \pi(\exp H) \in \pi(A)$ so that $d\pi(H)$ are real diagonal operators. Let $H_1, H_2 \in \mathfrak{a}_+$ such that $h_1 = \exp H_1, h_2 = \exp H_2 \in A_+$. Then

$$\pi(h_1)\pi(h_2) = \exp d\pi(H_1) \exp d\pi(H_2) = \exp d\pi(H_1 + H_2)$$

since $\mathfrak{a}$ is abelian and $d\pi$ respects the bracket. Notice that $|\pi(h_1)\pi(h_2)|$ is the exponent of the largest diagonal entry of the diagonal operator $d\pi(H_1) + d\pi(H_2)$. To arrive at $|\pi(h_1 h_2)| = |\pi(h_1)|\,|\pi(h_2)|$, it is sufficient to show that the sum of the largest diagonal entries $d\pi(H_1)$ and $d\pi(H_2)$ is also a diagonal entry of $d\pi(H_1 + H_2)$. To this end, we will use the theory of highest weights [14, p.108] on the finite dimensional irreducible representations of the complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (since $\mathfrak{g}$ is semisimple).

Let

$$\mathfrak{g} = (\mathfrak{a} \oplus \mathfrak{m}) \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha$$

be the restricted root decomposition of $\mathfrak{g}$ [11, p.263], where $\mathfrak{m}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$ and Σ is the set of restricted roots of $(\mathfrak{g}, \mathfrak{a})$. Let $\mathfrak{h}$ be the maximal abelian subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{h} \cap \mathfrak{p}$ and we set $\mathfrak{h}_{\mathfrak{k}} := \mathfrak{h} \cap \mathfrak{k}$. It is known that $\mathfrak{h}_{\mathbb{C}} := \mathfrak{h} \oplus i\mathfrak{h}$, the complexification of $\mathfrak{h}$, is a Cartan subalgebra of the complex semisimple $\mathfrak{g}_{\mathbb{C}}$ [11, p.259]. Let Δ be the set of roots of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ and set $\mathfrak{h}_{\mathbb{R}} := \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha$, where $H_\alpha \in \mathfrak{h}_{\mathbb{C}}$ is defined by the restriction to $\mathfrak{h}_{\mathbb{C}}$ of the Killing form, i.e., $B(H_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{h}_{\mathbb{C}}$. Then $\mathfrak{h}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{R}} \oplus i\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}} = \mathfrak{a} \oplus i\mathfrak{h}_{\mathfrak{k}}$. Each root $\alpha \in \Delta$ is real-valued on $\mathfrak{h}_{\mathbb{R}}$ [11, p.170]. Let $\Delta_{\mathfrak{p}} \subset \Delta$ be the set of roots which do not vanish identically on $\mathfrak{a}$. It is known that Σ is the set of restrictions of $\Delta_{\mathfrak{p}}$ to $\mathfrak{a}$ [11, p.263]. Furthermore we can choose a positive root system $\Delta^+ \subset \Delta$ so that $\mathfrak{a}_+$ is in the corresponding Weyl chamber (in $\mathfrak{h}_{\mathbb{R}}$) [21, p.431], that is, $\alpha(H) \geq 0$ for all $H \in \mathfrak{a}_+$, $\alpha \in \Delta^+$. So any root of Δ^+ restricts to either zero or an element in Σ^+ as a linear functional on $\mathfrak{a}$ [11, p.263].

The diagonal entries of the diagonal operator $d\pi(H)$, $H \in \mathfrak{a}_+ \subset \mathfrak{h}_{\mathbb{C}}$ are the eigenvalues of $d\pi(H)$ so that they are of the form $\mu(H)$, where μ are the weights of the representation $d\pi$ of $\mathfrak{g}_{\mathbb{C}}$ [14, p.107-108]. Let $\lambda \in \mathfrak{h}'$ be the highest weight of $d\pi$, where $\mathfrak{h}'$ denotes the dual space of $\mathfrak{h}$. From the theory of representation $\lambda - \mu$ is a sum of positive roots, i.e.,

$$\lambda - \mu = \sum_{\alpha \in \Delta^+} k_\alpha \alpha, \qquad k_\alpha \in \mathbb{N}.$$

Since the restrictions of the positive roots in Δ^+ to $\mathfrak{a}$ are either zero or elements in Σ^+, we conclude $\lambda(H) \geq \mu(H)$ for all $H \in \mathfrak{a}_+$. Since $\mathfrak{a}_+$ is a cone, $H_1 + H_2 \in \mathfrak{a}_+$. Thus $\lambda(H_1+H_2) = \lambda(H_1)+\lambda(H_2)$ is the largest diagonal entry (eigenvalue) of the diagonal operator $d\pi(H_1+H_2)$ and $\lambda(H_1)$, and $\lambda(H_2)$ are the largest diagonal entries (eigenvalues) of $d\pi(H_1)$ and $d\pi(H_2)$, respectively. ∎

The following theorem is an extension of Theorem 2.1.

Theorem 4.3. The following are equivalent and are valid.

$$\begin{aligned} a_+(a^t g b^{1-t}) &\prec [a_+(ag)]^t [a_+(gb)]^{1-t}, \ 0 \leq t \leq 1, \ a, b \in P, \ g \in G, && (7)\\ a_+(a^t g b^t) &\prec [a_+(g)]^{1-t} [a_+(agb)]^t, \ 0 \leq t \leq 1, \ a, b \in P, \ g \in G, && (8)\\ a_+(a^* g b) &\prec [a_+(aa^* g)]^{1/2} \, [a_+(gbb^*)]^{1/2}, \quad a, b, g \in G. && (9) \end{aligned}$$

Proof. We will first establish (7) and then the equivalence among the relations. Let $g \in G$ and write $g = k_1 a_+(g) k_2$, where $a_+(g) \in A_+$, $k_1, k_2 \in K$. Let π be any representation of G. Since the elements of $d\pi(\mathfrak{k})$ are skew Hermitian,

$$\|\pi(g)\| = \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1)\pi(a_+(g))\pi(k_2)\| = \|\pi(a_+(g))\|.$$

Since the spectral norm $\|\cdot\|$ is invariant under unitary equivalence, and $\|X\| = |X|$ for each positive definite operator X, $\|\pi(a_+(g))\| = |\pi(a_+(g))|$ and thus

$$\|\pi(g)\| = \|\pi(a_+(g))\| = |\pi(a_+(g))|. \tag{10}$$

Suppose $0 \leq t \leq 1$. Since the elements of $d\pi(\mathfrak{p})$ are Hermitian operators, $\pi(a)$ and $\pi(b)$ are positive definite operators,

$$\begin{aligned} |\pi(a_+(a^t g b^{1-t}))| &= \|\pi(a^t g b^{1-t})\| && \text{by (10)}\\ &= \|\pi^t(a)\pi(g)\pi^{1-t}(b)\|\\ &\leq \|\pi(a)\pi(g)\|^t \|\pi(g)\pi(b)\|^{1-t} && \text{by (2)}\\ &= \|\pi(ag)\|^t \|\pi(gb)\|^{1-t}\\ &= |\pi(a_+(ag))|^t \, |\pi(a_+(gb))|^{1-t} && \text{by (10)} \end{aligned}$$

The elements of $\pi(A)$ are positive diagonal operators under an appropriate orthonormal basis. Since $a_+^t(ag), a_+^{1-t}(gb) \in A_+$, $|\pi(a_+(ag))|^t = |\pi(a_+^t(ag))|$ and $|\pi(a_+(gb))|^{1-t} = |\pi(a_+^{1-t}(gb))|$. So

$$\begin{aligned} |\pi(a_+(ag))|^t \, |\pi(a_+(gb))|^{1-t} &= |\pi(a_+^t(ag))| \, |\pi(a_+^{1-t}(gb))|\\ &= |\pi(a_+^t(ag) \, \pi(a_+^{1-t}(gb))| && \text{by Lemma 4.2}\\ &= |\pi(a_+^t(ag) a_+^{1-t}(gb))|. \end{aligned}$$

As a result, $|\pi(a_+(a^t g b^{1-t}))| \le |\pi(a_+^t(ag)a_+^{1-t}(gb))|$ for any representation π of G. By Theorem 3.3 we have (7).

(7) $\Rightarrow$ (8): If $0 \le t \le 1$, then $0 \le 1-t \le 1$. If $a, b \in P$, so are their inverses. From (7)

$$\begin{aligned} a_+(a^t g b^t) &= a_+((a^{-1})^{1-t} a g b^{1-(1-t)}) \\ &\prec a_+^{1-t}(a^{-1}ag)a_+^t(agb) \\ &= a_+^{1-t}(g)a_+^t(agb), \end{aligned}$$

i.e., (8) is established.

(8) $\Rightarrow$ (9): Let $a, b \in G$. Write $a^* = kp$, $b^* = k'p'$ according to their Cartan decompositions. Then $b = p'k'^{-1}$. By (8) with $t = 1/2$,

$$\begin{aligned} a_+(a^* g b) &= a_+(kpgp'k'^{-1}) \\ &= a_+(pgp') \\ &= a_+((p^{-2})^{1/2}(p^2 g)(p'^2)^{1/2}) \\ &\prec a_+^{1/2}(p^2 g)a_+^{1/2}(p^{-2}p^2 g p'^2) \\ &= a_+^{1/2}(p^2 g)a_+^{1/2}(g p'^2). \end{aligned}$$

Since $aa^* = p^2$ and $bb^* = p'^2$, (9) follows.

(9) $\Rightarrow$ (7): Let $a, b \in P$. For $t = 0, 1$, (7) is trivial and for $t = 1/2$, it follows from (9). We will prove by induction for all $t = \frac{k}{2^n}$, where $k = 0, 1, \ldots, 2^n$ [3]. Let $t = \frac{2k+1}{2^n}$. Then $t = s + \rho$, where $s = \frac{k}{2^{n-1}}$ and $\rho = \frac{1}{2^n}$. Suppose that (7) is valid for all dyadic rationals with denominator 2^{n-1}. Then by induction and (9), with $\lambda := s + 2\rho$, we have

$$\begin{aligned} a_+(a^t g b^{1-t}) &= a_+(a^\rho(a^s g b^{1-\lambda})b^\rho) \\ &\prec a_+^{1/2}(a^{2\rho}a^s g b^{1-\lambda})a_+^{1/2}(a^s g b^{1-\lambda}b^{2\rho}) \\ &= a_+^{1/2}(a^\lambda g b^{1-\lambda})a_+^{1/2}(a^s g b^{1-s}) \\ &\prec a_+^{\lambda/2}(ag)a_+^{(1-\lambda)/2}(gb)a_+^{s/2}(ag)a_+^{(1-s)/2}(gb) \\ &= a_+^{(\lambda+s)/2}(ag)a_+^{1-(\lambda+s)/2}(gb) \\ &= a_+^t(ag)a_+^{1-t}(gb). \end{aligned}$$

The general case follows from continuity of the spectral radius and Theorem 3.3. ■

Furuta's inequality [8] asserts that if $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite, then

$$\|A^t B^t\| \le \|AB\|^t, \qquad 0 \le t \le 1. \tag{11}$$

It is equivalent to say that

$$\|A^s B^s\| \ge \|AB\|^s, \qquad s \ge 1. \tag{12}$$

See [1, 7, 24]. We have the following extension of Furuta's inequality.

Corollary 4.4. Let $a, b \in P$. Then

1. $a_+(a^t b^t) \prec a_+^t(ab)$, $0 \le t \le 1$.
2. $a_+^t(ab) \prec a_+(a^t b^t)$, $t \ge 1$.

Hence $\varphi(t) = [a_+(a^{1/t} b^{1/t})]^t$ is a decreasing function on $t > 0$ with respect to the partial order $\prec$, i.e., $\varphi(s) \prec \varphi(t)$ if $s \ge t > 0$.

Proof. By setting g to be the identity in (8) we have $a_+(a^t b^t) \prec a_+^t(ab)$, $0 \le t \le 1$. When $t \ge 1$, $a_+(a^{1/t} b^{1/t}) \prec a_+^{1/t}(ab)$. Then replace a, b by a^t and b^t respectively to have $a_+(ab) \prec a_+^{1/t}(a^t b^t)$. Let $s \ge t > 0$. Then $s/t > 1$ and

$$a_+(a^s b^s) = a_+((a^t)^{s/t} (b^t)^{s/t}) = a_+^{s/t}(a^t b^t)$$

so that $\varphi(t)$ is decreasing on $t > 0$. ■

Corollary 4.5. *For* $f, g \in G$, $a_+(fg) \prec a_+(f) a_+(g)$.

Proof. By (9),

$$a_+(fg) \prec a_+^{1/2}(f^* f) a_+^{1/2}(g g^*).$$

Use Lemma 4.1 to obtain $a_+(fg) \prec a_+(f) a_+(g)$. ■

Remark 4.6. When $G = \mathrm{GL}_n(\mathbb{C})$, by Corollary 4.5 the singular values of a product AB is log majorized by the product of the singular values of $A, B \in \mathbb{C}_{n\times n}$, assuming that singular values are all arranged in descending order.

Nakamoto [9] showed that (12) holds for normal matrices $A, B \in \mathbb{C}_{n\times n}$ and natural numbers s. An element $g \in G$ is said to be *normal* if $gg^* = g^* g$. It is equivalent to say that $kp = pk$, where $g = kp$ is the Cartan decomposition of g. Since Cartan decomposition is unique up to conjugation [11, p.183], normality is independent of the choice of K and P. Clearly the elements of P are normal. Normality is reduced to the usual normality when $G = \mathrm{SL}_n(\mathbb{C})$. Now we extend Nakamoto's result.

Corollary 4.7. Let $f, g \in G$ be normal. Then $a_+^n(fg) \prec a_+(f^n g^n), n \in \mathbb{N}$.

Proof. Let $f = kp$, $g = k'p'$ be the Cartan decompositions of $f, g \in G = KP$. Since f, g are normal, we have $kp = pk$ and $k'p' = p'k'$. Then

$$a_+^n(fg) = a_+^n(kpk'p') = a_+^n(kpp'k') = a_+^n(pp').$$

By Corollary 4.4,

$$a_+^n(pp') \prec a_+(p^n p'^n) = a_+(k^n p^n p'^n k'^n) = a_+((kp)^n (k'p')^n) = a_+(f^n g^n).$$

■

5. Inequalities for hyperbolic components

Furuta's inequality (11) is equivalent to the following inequality: for any positive semi-definite $A, B \in \mathbb{C}_{n\times n}$,

$$\lambda_1(A^tB^t) \le \lambda_1^t(AB), \qquad 0 \le t \le 1. \tag{13}$$

One can deduce the equivalence by

$$|AB| = \lambda_1(AB), \tag{14}$$

where $\lambda_1(AB)$ is the largest eigenvalue of the matrix AB whose eigenvalues are those of the positive semi-definite $B^{1/2}AB^{1/2}$.

Inequality (13) concerns about the largest eigenvalues of AB and A^tB^t when $A, B \in \mathbb{C}_{n\times n}$ are positive semi-definite. Since the eigenvalues of AB are nonnegative, (13) can be viewed as a result on the largest eigenvalue modulus. So we will consider the *hyperbolic component* of $g \in G$ of a semisimple connected noncompact Lie group G for an appropriate extension.

An element $X \in \mathfrak{g}$ is called *real semisimple* if $\mathrm{ad}\, X \in \mathrm{End}\,(\mathfrak{g})$ is diagonalizable over $\mathbb{R}$. It is equivalent to say that $\mathrm{ad}\,(X)$ is diagonalizable over $\mathbb{C}$ and the eigenvalues of $\mathrm{ad}\,(X)$ are real. An element $X \in \mathfrak{g}$ is called nilpotent if $\mathrm{ad}\, X$ is nilpotent. An element $g \in G$ is called hyperbolic if $g = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple and is called unipotent if $g = \exp(X)$ where $X \in g$ is nilpotent. An element $g \in G$ is elliptic if $\mathrm{Ad}(g) \in \mathrm{Aut}\,(\mathfrak{g})$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The *complete multiplicative Jordan decomposition* [21, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as $g = ehu$, where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write

$$g = e(g)h(g)u(g).$$

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to a unique element $b(h) \in A_+$ [21, Proposition 2.4]. Denote

$$b(g) := b(h(g)).$$

It is known that [21, Proposition 6.2] P^2 is the set of all hyperbolic elements and $b(g) \prec a_+(g)$ for all $g \in G$ [21, Theorem 5.4].

Example 5.1. When $G = \mathrm{SL}_n(\mathbb{C})$, $g = ehu$ is the usual complete multiplicative Jordan decomposition [11, p.431] and

$$b(g) = \mathrm{diag}\,(|\lambda_1|, \ldots, |\lambda_n|),$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of g with descending moduli.

Lemma 5.2.

1. Let $f, g \in G$. Then $h(fg) = g^{-1}h(gf)g$ and $b(fg) = b(gf)$.
2. If $f \in P$, then $b(f) = a_+(f)$. So $a_+(g^*g) = a_+(gg^*) = b(g^*g) = b(gg^*)$ for all $g \in G$.
3. Let $f, g \in P$. Then $b(f^2g^2) = a_+^2(fg) = a_+^2(gf)$.

Proof. (1) Let $fg = ehu$ be the CMJD of fg where $f, g \in G$. Then $gf = g(fg)g^{-1} = (geg^{-1})(ghg^{-1})(gug^{-1})$. By the uniqueness of CMJD, $h(fg) = g^{-1}h(gf)g$ follows immediately. Now $b(fg) = b(h(fg)) = b(g^{-1}h(gf)g) = b(gf)$.

(2) Since P is K-conjugate to some element in A_+, $b(f) = a_+(f)$.

(3) By (1) $h(f^2g^2) = g^{-1}h(gf^2g)g$ so that

$$b(f^2g^2) = b(g^{-1}h(gf^2g)g) = b(gf^2g) = b((gf)(gf)^*).$$

The element $(gf)(gf)^*$ is in P so that by (2) and Lemma 4.1

$$b(f^2g^2) = b((gf)(gf)^*) = a_+((gf)(gf)^*) = a_+^2(fg).$$

■

Theorem 5.3. Let $a, b \in P$. The following are equivalent and are valid.

1. $a_+(a^tb^t) \prec a_+^t(ab)$, $0 \le t \le 1$.
2. $b(f^tg^t) \prec b^t(fg)$, $0 \le t \le 1$.

In other words, for any hyperbolic element $\ell \in G$, if we write $\ell = fg$, where $f, g \in P$, then $b(f^tg^t) \prec b^t(\ell)$, $0 \le t \le 1$.

Proof. Statement (1) is Corollary 4.4 (1). The set of all hyperbolic elements is P^2. Since $f, g \in P$, $f^t, g^t \in P$ and thus $fg, f^tg^t \in P^2$ are hyperbolic for all $t \in \mathbb{R}$. By Lemma 5.2 and Corollary 4.4

$$b^{1/2}(f^{2t}g^{2t}) = a_+(f^tg^t) \prec a_+^t(fg) = b^{t/2}(f^2g^2).$$

So $b(f^{2t}g^{2t}) \prec b^t(f^2g^2)$. Then replace f^2 and g^2 by f and g, respectively, to obtain the desired result.

Conversely, suppose that $b(f^tg^t) \prec b^t(fg)$ for all $0 \le t \le 1$, where $f, g \in P$. Then

$$a_+(f^tg^t) = b^{1/2}(f^{2t}g^{2t}) \prec b^{t/2}(f^2g^2) = a_+^t(fg).$$

■

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