Integr. equ. oper. theory 99 (9999), 1–6
0378-620X/99000-0, DOI 10.1007/s00020-003-0000

Integral Equations and Operator Theory

λ-Aluthge iteration and spectral radius

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Abstract. Let $T \in B(H)$ be an invertible operator on the complex Hilbert space H. For $0 < \lambda < 1$, we extend Yamazaki's formula of the spectral radius in terms of the λ-Aluthge transform $\Delta_\lambda(T) := |T|^\lambda U|T|^{1-\lambda}$ where $T = U|T|$ is the polar decomposition of T. Namely, we prove that $\lim_{n\to\infty} \| \Delta_\lambda^n(T) \| = r(T)$ where $r(T)$ is the spectral radius of T and $\| \cdot \|$ is a unitarily invariant norm such that $(B(H), \| \cdot \|)$ is a Banach algebra with $\| I \| = 1$.

1. Introduction

Let H be a complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on H. For $0 < \lambda < 1$, the λ-Aluthge transform of T [2, 5, 11, 15] is

$$\Delta_\lambda(T) := |T|^\lambda U|T|^{1-\lambda},$$

where $T = U|T|$ is the polar decomposition of T, that is, U is a partial isometry and $|T| = (T^*T)^{1/2}$. Set $\Delta_\lambda^n(T) := \Delta_\lambda(\Delta_\lambda^{n-1}(T))$, $n \geq 1$ and $\Delta_\lambda^0(T) := T$. When $\lambda = \frac{1}{2}$, it is called the Althuge transform [1] of T. See [3, 4, 7, 8, 11, 12, 13, 19]. Yamazaki [18] established the following interesting result

$$\lim_{n\to\infty} \|\Delta_{\frac{1}{2}}^n(T)\| = r(T), \tag{1.1}$$

where $r(T)$ is the spectral radius of T and $\|T\|$ is the spectral norm of T. Wang [17] then gave an elegant simple proof of (1.1) but apparently there is a gap. See Remark 2.6 for details and Remark 2.9 for a fix. Clearly

$$\|\Delta_\lambda(T)\| \leq \|T\| \tag{1.2}$$

and thus $\{\|\Delta_\lambda^n(T)\|\}_{n\in\mathbb{N}}$ is nonincreasing. Since the spectra of T and $\Delta_\lambda(T)$ are identical [11, 5],

$$r(\Delta_\lambda(T)) = r(T). \tag{1.3}$$

2000 Mathematics Subject Classification. Primary 47A30, 47A63
In memory of my brother-in-law, Johnny Kei-Sun Man, who passed away on January 16, 2008, at the age of fifty nine.

Antezana, Massey and Stojanoff [5] proved that for any square matrix X,

$$\lim_{n\to\infty} \|\Delta_\lambda^n(X)\| = r(X). \tag{1.4}$$

When T is invertible, the polar decomposition $T = U|T|$ is unique [16, p.315] in which U is unitary and $|T|$ is invertible positive so that $\Delta_\lambda(T)$ is also invertible. Our goal is to show that (1.4) is true for invertible operator $T \in B(H)$ and unitarily invariant norm under which $B(H)$ is a Banach algebra with $\|| I \|| = 1$ [16, p.227-228], by extending the ideas in [17].

2. Main results

The following inequalities are known as Heinz's inequalities [10, 14].

Lemma 2.1. (Heinz) Let $\|| \cdot \||$ be a unitarily invariant norm on $B(H)$. For positive $A, B \in B(H)$, $X \in B(H)$ and $0 \le \alpha \le 1$,

$$\|| A^\alpha X B^{1-\alpha} \|| \le \|| AX \||^{\alpha} \|| XB \||^{1-\alpha} \tag{2.1}$$

$$\|| A^\alpha X B^{\alpha} \|| \le \|| AXB \||^{\alpha} \|| X \||^{1-\alpha}. \tag{2.2}$$

Remark 2.2. In [17] inequality (2.2) and McIntosh inequality

$$\|| A^* X B \|| \le \|| AA^* X \||^{1/2} \|| XBB^* \||^{1/2}$$

are used in the proof of (1.1). Yamazaki's proof [18] uses (2.2). We note that (2.2) and McIntosh's inequality follow from (2.1).

We simply write $T_0 := T$ and $T_n := \Delta_\lambda^n(T)$, $n \in \mathbb{N}$, once we fix $0 \le \lambda \le 1$.

Lemma 2.3. Let $T \in B(H)$ and $0 \le \lambda \le 1$. Let $\|| \cdot \||$ be a unitarily invariant norm on $B(H)$. Then for $k, n \in \mathbb{N}$,

$$\|| (T_{n+1})^k \|| \le \|| (T_n)^k \||. \tag{2.3}$$

So for each $k \in \mathbb{N}$, $\lim_{n\to\infty} \|| (T_n)^k \||$ exists. Moreover if T is invertible, then

$$\|| (T_{n+1})^k |T_n|^{2\lambda-1} \|| \le \|| (T_n)^{k+1} \||^{\lambda} \|| (T_n)^{k-1} \||^{1-\lambda}, \tag{2.4}$$

$$\|| |T_n|^{1-2\lambda} (T_{n+1})^k \|| \le \|| (T_n)^{k+1} \||^{1-\lambda} \|| (T_n)^{k-1} \||^{\lambda}. \tag{2.5}$$

Proof. Denote by $T_n = U_n T_n$ the polar decomposition of T_n. Since $T_{n+1} = |T_n|^\lambda U_n |T_n|^{1-\lambda}$,

$$(T_{n+1})^k = |T_n|^\lambda (T_n)^{k-1} U_n |T_n|^{1-\lambda}. \tag{2.6}$$

By (2.1)

$$\|| (T_{n+1})^k \|| \le \|| |T_n| (T_n)^{k-1} U_n \||^{\lambda} \|| (T_n)^{k-1} U_n |T_n| \||^{1-\lambda} = \|| (T_n)^k \||$$

since $\|| \cdot \||$ is unitarily invariant. So (2.3) is established.

Suppose that T is invertible so that $|T_n|^{2\lambda-1}$ exists for $0 \le \lambda \le 1$. From (2.6)

$$(T_{n+1})^k |T_n|^{2\lambda-1} = |T_n|^\lambda (T_n)^{k-1} U_n |T_n|^\lambda.$$

Since $|||\cdot|||$ is unitarily invariant and U_n is unitary, by (2.2)

$$\begin{aligned} |||(T_{n+1})^k|T_n|^{2\lambda-1}||| &\le |||\,|T_n|(T_n)^{k-1}U_n|T_n|\,|||^{\lambda}\, |||(T_n)^{k-1}U_n|||^{1-\lambda} \\ &= |||(T_n)^{k+1}|||^{\lambda}\, |||(T_n)^{k-1}|||^{1-\lambda}. \end{aligned}$$

Similarly (2.5) is established. □

When $\lambda = 1/2$, $|T_n|^{2\lambda-1} = I$ for any $T \in B(H)$. But the above computation does not work without assuming that T is invertible since the polar decomposition of a general $T \in B(T)$ only yields $T = U|T|$ where U is only a partial isometry [16, p.316]. The spectral norm enjoys $\|T\| = \|\,|T|\,\|$ which is not valid for general unitarily invariant norm $|||\cdot|||$.

Lemma 2.4. (Spectral radius formula) [16, p.235] Suppose that $B(H)$ is a Banach algebra with respect to the norm $|||\cdot|||$ (not necessarily unitarily invariant). For $T \in B(H)$,

$$r(T) = \lim_{k\to\infty} |||T^k|||^{1/k} = \inf_{k\in\mathbb{N}} |||T^k|||^{1/k}.$$

In particular $|||T||| \ge r(T)$.

For $B(H)$ to be a Banach algebra with respect to $|||\cdot|||$, the norm in Lemma 2.4 has to be submultiplicative, i.e., $|||ST||| \le |||S|||\,|||T|||$ [16, p.227]. The unitarily invariant norm $|||\cdot|||$ in Lemma 2.3 need not be so. The condition $|||I||| = 1$ is inessential for the formula $r(T) = \lim_{k\to\infty} |||T^k|||^{1/k}$, i.e, it is still valid even $|||I||| > 1$. The formula $r(T) = \inf_{k\in\mathbb{N}} |||T^k|||^{1/k}$ is valid for any normed algebra [6, p.236]).

Lemma 2.5. Suppose that $B(H)$ is a Banach algebra with respect to the unitarily invariant norm $|||\cdot|||$ and $|||I||| = 1$. Let $0 < \lambda < 1$. Let $s_k := \lim_{n\to\infty} |||(T_n)^k|||$ and $s := s_1$. If $T \in B(H)$ is non-quasinilpotent, then $s > 0$. Moreover if $T \in B(H)$ is invertible, then $s_k = s^k$ for each $k \in \mathbb{N}$.

Proof. Let $T \in B(H)$. By (2.3), for each $k \in \mathbb{N}$, the sequence $\{|||(T_n)^k|||\}_{n\in\mathbb{N}}$ is nonincreasing so that $s_k := \lim_{n\to\infty} |||(T_n)^k|||$ exists. By Lemma 2.4 and (1.3) $|||T_n||| \ge r(T_n) = r(T)$. The spectrum $\sigma(T)$ of T is a compact nonempty set. If T is non-quasinilpotent, i.e., $r(T) > 0$ so that

$$s := s_1 = \lim_{n\to\infty} |||T_n||| \ge r(T) > 0. \tag{2.7}$$

Now assume that T is invertible. We proceed by induction to show that $s_k = s^k$ for all $k \in \mathbb{N}$. When $k = 1$, the statement is trivial. Suppose that the statement is true for $1 \le k \le m$.

Case 1: $0 < \lambda \le 1/2$. By (2.1) ($A = |T_n|$, $X = B = I$) we have

$$0 < |||\,|T_n|^{1-2\lambda}||| \le |||\,|T_n|\,|||^{1-2\lambda}$$

since $\|\!| I \|\!| = 1$ and $0 \le 1-2\lambda < 1$. Since T is invertible, $|T_n|$ is also invertible and thus $|T_n|^{2\lambda-1}$ exists. So

$$\begin{aligned}
\frac{\|\!| (T_{n+1})^m \|\!|}{\|\!| |T_n| \|\!|^{1-2\lambda}} &\le \frac{\|\!| (T_{n+1})^m \|\!|}{\|\!| |T_n|^{1-2\lambda} \|\!|} \\
&\le \|\!| (T_{n+1})^m |T_n|^{2\lambda-1} \|\!| \qquad \text{since } \|\!| \cdot \|\!| \text{ is submultiplicative} \\
&\le \|\!| (T_n)^{m+1} \|\!|^{\lambda} \|\!| (T_n)^{m-1} \|\!|^{1-\lambda} \qquad \text{by (2.4)} \\
&\le \|\!| (T_n)^m \|\!|^{\lambda} \|\!| T_n \|\!|^{\lambda} \|\!| (T_n)^{m-1} \|\!|^{1-\lambda}.
\end{aligned}$$

By the induction hypothesis, taking limits as $n \to \infty$ yields

$$\frac{s^m}{s^{1-2\lambda}} \le s_{m+1}^{\lambda} s^{(m-1)(1-\lambda)} \le s^{m\lambda} s^{\lambda} s^{(m-1)(1-\lambda)},$$

where $s > 0$ by (2.7). We have $s^{(m+1)\lambda} \le s_{m+1}^{\lambda} \le s^{(m+1)\lambda}$ and hence $s_{m+1} = s^{m+1}$.

Case 2. $1/2 < \lambda < 1$. Similar to Case 1, by (2.1) we have

$$\|\!| |T_n|^{2\lambda-1} \|\!| \le \|\!| |T_n| \|\!|^{2\lambda-1}$$

and

$$\begin{aligned}
\frac{\|\!| (T_{n+1})^m \|\!|}{\|\!| |T_n| \|\!|^{2\lambda-1}} &\le \frac{\|\!| (T_{n+1})^m \|\!|}{\|\!| |T_n|^{2\lambda-1} \|\!|} \\
&\le \|\!| |T_n|^{1-2\lambda} T_{n+1}^m \|\!| \\
&\le \|\!| (T_n)^{m+1} \|\!|^{1-\lambda} \|\!| (T_n)^{m-1} \|\!|^{\lambda} \qquad \text{by (2.5)} \\
&\le \|\!| (T_n)^m \|\!|^{1-\lambda} \|\!| T_n \|\!|^{1-\lambda} \|\!| (T_n)^{m-1} \|\!|^{\lambda}.
\end{aligned}$$

So

$$\frac{s^m}{s^{2\lambda-1}} \le s_{m+1}^{1-\lambda} s^{(m-1)\lambda} \le s^{m(1-\lambda)} s^{(1-\lambda)} s^{(m-1)\lambda}$$

which leads to $s_{m+1} = s^{m+1}$. □

Remark 2.6. Suppose $\lambda = 1/2$. In the proof of [17, Lemma 4], the possibility that $s = 0$ is not considered (the spectral $\|\cdot\|$ is the norm under consideration). It amounts to $r(T) = 0$, that is, T is quasinilpotent [9, p.50], [13, p.381]. In the above induction proof, if $\lambda = 1/2$, one cannot deduce that $s_{m+1} = s^{m+1}$ for $\|\!| \cdot \|\!|$, granted that $s^m \le s_{m+1}^{1/2} s^{(m-1)/2} \le s^m$ (that relies on (2.4) which is under the assumption that T is invertible) is valid. However if $\|\!| \cdot \|\!| = \|\cdot\|$ and $\lambda = 1/2$ (the setting in [17]), then one has $s_k = s^k$ for all $k \in \mathbb{N}$ because $\|T\| = \| |T| \|$ for any $T \in B(H)$.

Theorem 2.7. Suppose that $B(H)$ is a Banach algebra with respect to the unitarily invariant norm $\|\!| \cdot \|\!|$ and $\|\!| I \|\!| = 1$. Let $T \in B(H)$ be invertible and $0 < \lambda < 1$. Then

$$\lim_{n\to\infty} \|\!| \Delta_\lambda^n(T) \|\!| = r(T). \tag{2.8}$$

Proof. From (2.3) and Lemma 2.5, for each $k \in \mathbb{N}$, the sequence $\{ \|\!| (T_n)^k \|\!|^{1/k}\}_{n\in\mathbb{N}}$ is nonincreasing and converges to $s := \lim_{n\to\infty} \|\!| T_n \|\!|$. So for all $n, k \in \mathbb{N}$,

$$s \le \|\!| (T_n)^k \|\!|^{1/k}.$$

Recall (2.7) $s \ge r(T)$. Suppose $r(T) < s$, that is, $r(T_n) < s$ (for all n). Then for a fixed $n \in \mathbb{N}$ and sufficiently large k, by Lemma 2.4, we would have

$$\|\!| (T_n)^k \|\!|^{1/k} < s,$$

a contradiction. So $r(T) = s$. □

We remark that (2.8) is not true if $\lambda = 0$ (since $\Delta_0^n(T) = T$ for all n so that $\|\!| \Delta_0^n(T) \|\!| = \|\!| T \|\!|$) or $\lambda = 1$ (since $\|\!| \Delta_1^n(T) \|\!| = \|\!| T \|\!|$ for all n). The condition $\|\!| I \|\!| = 1$ is essential, for example, if $\|\!| \cdot \|\!| = \alpha \| \cdot \|$ where $\alpha > 1$, then $\|\!| I \|\!| = \alpha$ but (2.7) is not valid.

Corollary 2.8. Let $T \in B(H)$ be invertible and $0 < \lambda < 1$. Then

$$\lim_{n\to\infty} \|\Delta_\lambda^n(T)\| = r(T).$$

Clearly (1.4) follows from Corollary 2.8 by the continuity of spectral radius and the continuity of $\Delta_{1/2}$ [5, Theorem 3.6] in the finite dimensional case. However, such continuity result is invalid for the infinite dimensional case [9, p.54]. Moreover the group of invertible operators is not dense [9, p.70].

We surmise that Theorem 2.7 is true for non-invertible $T \in B(T)$ as well.

Remark 2.9. Of course the statement in Corollary 2.8 is valid for any $T \in B(H)$ when $\lambda = 1/2$, i.e., (1.1). As pointed out in Remark 2.6, there is a gap in the proof in [17]. We now fill the gap. By Remark 2.6 $\lim_{n\to\infty} \|\Delta_{1/2}^n(T)\| = r(T)$ is valid for non-quasinilpotent $T \in B(H)$ as the proof of Theorem 2.7 works for non-quasinilpotent T (because (2.3) is valid for any $T \in B(H)$ and $s_k = s^k$ for non-quasinilpotent by Remark 2.6). If T is quasinilpotent, then consider the orthogonal sum $T \oplus cI \in B(H \oplus H)$. We may consider $T \neq 0$. Notice that $T \oplus cI$ is non-quasinilpotent if $c > 0$. Since $\Delta_{1/2}^n(T \oplus cI) = \Delta_{1/2}^n(T) \oplus \Delta_{1/2}^n(cI) = \Delta_{1/2}^n(T) \oplus cI$ and $r(T \oplus cI) = r(cI) = c$, by Remark 2.6,

$$\max\{\|\Delta_{1/2}^n(T)\|, c\} = \|\Delta_{1/2}^n(T) \oplus cI\| = \|\Delta_{1/2}^n(T \oplus cI)\| \to r(T \oplus cI) = c,$$

for any $c > 0$, as $n \to \infty$. Letting $c \to 0$ to yield $\lim_{n\to\infty} \|\Delta_{1/2}^n(T)\| = 0$.

Acknowledgement: The author is thankful to Ilya Spitkovsky for some helpful comments that lead to the improvement of the paper, in particular Remark 2.9.

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