Jordan Normal Form Revisited

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Let us start with some online conversations.

conversation 1

conversation 2
1. Similarity

Given $A, B \in \mathbb{C}_{n \times n}$. $A$ is said to be similar to $B$ if there is a nonsingular matrix $P$ such that $P^{-1}AP = B$, denoted by $B \sim A$.

**Fact:** Similarity is an equivalence relation

1. **Reflexive** ($A \sim A$): $A = IAI = I^{-1}AI$ for all $A \in \mathbb{C}_{n \times n}$.

2. **Symmetric** ($A \sim B$ implies $A \sim B$): $P^{-1}AP = B$ implies $PBP^{-1} = A$.

3. **Transitive** ($A \sim B$ and $B \sim C$ imply $C \sim A$) $P^{-1}AP = B$ and $Q^{-1}BQ = C$ imply $(PQ)^{-1}A(PQ) = C$.

As an equivalence relation, similarity partitions $\mathbb{C}_{n \times n}$ into equivalence classes $\rightarrow$ representative (normal form, canonical form)
2. Jordan Normal Form

Each $A \in \mathbb{C}_{n \times n}$ is similar to a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

where each block $J_i$ is a square matrix of the form (Jordan block)

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}_{n_i \times n_i}.$$

Remarks: The eigenvalues $\lambda_i$ and $\lambda_k$ may not be distinct for $i \neq k$. 
Example:

\[ A = \begin{bmatrix}
5 & 4 & 2 & 1 \\
0 & 1 & -1 & -1 \\
-1 & -1 & 3 & 0 \\
1 & 1 & -1 & 2
\end{bmatrix}. \]

\[ P^{-1}AP = J \] where

\[ J = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 4 & 1 \\
0 & 0 & 0 & 4
\end{bmatrix}, \quad P = \begin{bmatrix}
-1 & 1 & 1 & 1 \\
1 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 1 & 0
\end{bmatrix}. \]

Check: \[ AP = PJ. \]
A little History:
In 1870 the Jordan canonical form appeared in Treatise on substitutions and algebraic equations by Camille Jordan (1838-1922). It appears in the context of a canonical form for linear substitutions over the finite field of order a prime.

The Jordan of Gauss-Jordan elimination is Wilhelm Jordan (1842 to 1899).

Jordan algebras are called after the German physicist and mathematician Pas- cual Jordan (1902 to 1980).
3. Numerical unstable

Consider

\[ A = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}. \]

If \( \epsilon = 0 \), then the Jordan normal form is simply \( \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \). However, for \( \epsilon \neq 0 \), the Jordan normal form is

\[ \begin{bmatrix} 1 + \sqrt{\epsilon} & 0 \\ 0 & 1 - \sqrt{\epsilon} \end{bmatrix}. \]

So it is hard to develop a robust numerical algorithm for the Jordan normal form. For this reason, the Jordan normal form is usually avoided in numerical analysis.

Matlab command \([P, J] = \text{jordan}(A)\)
4. Applications

Remark: Jordan normal form may be useless for *numerical* linear algebra, it has a valid place in *applied* linear algebra.


(1) (a) Every $A \in \mathbb{C}_{n \times n}$ is similar to a complex symmetric matrix.

(b) Every $A \in \mathbb{C}_{n \times n}$ is a product of two complex symmetric matrices; one of which can be chosen nonsingular.
(2) $A^m \to 0$ if and only if the eigenvalue moduli of $A$ are less than 1.

**Reason:** It suffices to consider $J = \begin{bmatrix} \lambda & 1 \\ & \ddots \\ & & \lambda \end{bmatrix} = \lambda I + N \in \mathbb{C}_{k \times k}$.

Since $N^m = 0$ for all $m \geq k$, we have

$$J^m = (\lambda I + N)^m = \sum_{i=0}^{m} \binom{m}{i} \lambda^i N^{m-i} = \sum_{i=m-k+1}^{m} \binom{m}{i} \lambda^i N^{m-i}$$

(a) Since the diagonal entries of $J^m$ are $\lambda^m$, if $J^m \to 0$, then $\lambda^m \to 0$, i.e., $|\lambda| < 1$.

(b) Conversely, if $|\lambda| < 1$, then

$$\left| \left( \begin{array}{c} m \\ m-j \end{array} \right) \lambda^{m-j} \right| = \left| \frac{m(m-1)(m-2) \cdots (m-j+1)\lambda^m}{j!\lambda^j} \right| \leq \left| \frac{m^j \lambda^m}{j!\lambda^j} \right| \to 0$$

as $m \to \infty$ by l’Hopital’s rule.

It finds application in population growth.
(3) $A$ is similar to its transpose $A^T$.

By Jordan normal form $A = PJP^{-1}$. So $A \sim A^T$ amounts to $J \sim J^T$. Now

$$
\begin{bmatrix}
\lambda & & \\
& \ddots & \\
& & 1
\end{bmatrix}
= 
\begin{bmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{bmatrix}
\begin{bmatrix}
\lambda & & \\
& \ddots & \\
& & 1
\end{bmatrix}
\begin{bmatrix}
1 & & \\
& \ddots & \\
& & 1
\end{bmatrix}
$$

(4) Solution to the a linear system of ODE:

$$
x'(t) = Ax(t), \quad x(0) = x_0
$$

$$
x(t) = Pe^{tJ}P^{-1}x_0 = P\begin{bmatrix}e^{tJ_1} & & \\
& \ddots & \\
& & e^{tJ_p}\end{bmatrix}P^{-1}x_0,
$$

where

$$
e^{tJ_i} = e^{t\lambda_i} 
\begin{bmatrix}
1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-1}}{(n-1)!} \\
& 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\
& & 1 & \cdots & \\
& & & \ddots & 1
\end{bmatrix}
$$
5. **Proofs**


8. .............
6. Basis change

We switch to the lower triangular version of Jordan normal form: Each $A \in \mathbb{C}_{n \times n}$ is similar to a block diagonal matrix, i.e., for some nonsingular $P$

$$P^{-1}AP = J = \begin{bmatrix} J_1 & \cdots \\ \cdots & \cdots \\ & & J_p \end{bmatrix}$$

where each block $J_i$ is a square matrix of the form (Jordan block)

$$J_i = \begin{bmatrix} \lambda_i \\ 1 & \lambda_i \\ \cdots & \cdots \\ & & \cdots \\ & & 1 & \lambda_i \end{bmatrix} \in \mathbb{C}_{n_i \times n_i}.$$

An interpretation: $P^{-1}AP$ is the matrix representation with respect to a new basis given by the columns of $P$, since $P^{-1}(\cdot)P$ means a change of basis.
Partition $P = [P_1 | \cdots | P_p]$ accordingly. Let $P_i = [v_1 | v_2 | \cdots | v_{n_i}] \in \mathbb{C}^{n \times n_i}$.

Jordan form amounts to

$$AP_i = P_iJ_i = P_i(\lambda_iI + N_i), \quad N_i = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

From $AP_i = P_i(\lambda_iI + N_i)$ we have

$$[Av_1 \mid Av_2 \mid \cdots \mid Av_{n_i}] = \lambda_i[v_1 \mid v_2 \mid \cdots \mid v_{n_i}] + [v_2 \mid v_3 \mid \cdots \mid v_{n_i} \mid 0],$$

i.e.

$$[(A - \lambda_iI)v_1 \mid (A - \lambda_iI)v_2 \mid \cdots \mid (A - \lambda_iI)v_{n_i}] = [v_2 \mid v_3 \mid \cdots \mid v_{n_i} \mid 0].$$

In other words, $v_1, v_2, \ldots, v_{n_i}$ are related: set $v := v_1$, $m := n_i$, $\lambda := \lambda_i$,

$$v_1 = v, v_2 = (A - \lambda I)v, \ldots, v_m = (A - \lambda I)^{m-1}v \quad (\text{chain})$$
7. **A Short proof**


In terms of the language of linear operator:

**Theorem 7.1.** Let \( T : V \rightarrow V \) be a linear operator acting on a finite dimensional space over \( \mathbb{C} \). Then \( V \) has a \( T \)-Jordan basis, that is, an ordered basis which consists of Jordan sequences: a \((T, \lambda)\)-Jordan sequences, where \( \lambda \) is a scalar, is a sequence of vectors of the form

\[
v, (T - \lambda I)v, \ldots, (T - \lambda I)^{m-1}v
\]

for \( v \in V \) and \( m \geq 1 \) such that \((T - \lambda I)^m v = 0\).
We will use **contradiction**:

1. Assuming that the theorem is false, let $T : V \to V$ be a counterexample with $\dim V \geq 2$ **minimal**, thus $V \neq 0$.
2. $T$ has an eigenvalue $\mu$ in $\mathbb{C}$. Replacing $T$ by $T - \mu I$ we may assume that $\mu = 0$. Thus
   \[ \dim T(V) < \dim V. \]
3. By the minimality of $\dim V$, $T(V)$ has a $T_0$-**Jordan basis**, where $T_0$ is the restriction of $T$ to $T(V)$, i.e.,
   \[ T_0 = T|_{T(V)} : T(V) \to T(V), \quad T_0(w) = T(w) \]
4. Let $V'$ be a subspace of maximal dimension among all subspaces of $V$
   (a) invariant under $T$, and
   (b) with a **Jordan basis**, $B$, containing a basis of $T(V)$.
   Clearly $T(V) \subset V' \subset V$. 
(5) Claim: \(T(V) = T(V').\) Clearly \(T(V') \subset T(V)\) since \(V' \subset V.\)

To prove \(T(V) \subset T(V')\) we will show \(B \cap T(V) \subset T(V').\)

Indeed if \(w \in B \cap T(V),\) then \(w\) belongs to a \((T, \lambda)\)-Jordan sequence

\[ u, (T - \lambda I)u, \ldots, (T - \lambda I)^{m-1}u \in B \subset V'. \]

Case 1: \(\lambda = 0.\) If \(w \neq u\) (not the first one), then \(w = T^k u \in T(V'),\)
\(1 \leq k \leq m.\) If \(w = u,\) pick any \(w' \in V\) such that \(T(w') = w\) since \(w \in B \cap T(V) \subset T(V).\) If \(w' \notin V',\) then we may add \(w'\) to the above sequences thus extending \(B\) to a Jordan basis of a subspace properly containing \(V',\) a contradiction. Hence \(w' \in V',\) i.e., \(w \in T(V').\)

Case 2: \(\lambda \neq 0.\) Observation: for \(v \in V'\)

\[ v \in T(V') \iff (T - \lambda I)v = Tv - \lambda v \in T(V'). \]

Since \(w \in B \subset V'\) and \((T - \lambda I)^m w = 0 \in T(V')\) we see that

\[ (T - \lambda I)^{m-1}w \in T(V'), \quad (T - \lambda I)^{m-2}w \in T(V'), \quad \ldots, w \in T(V'). \]

Since \(B\) contains a basis of \(T(V),\) we have \(T(V) = T(V').\)
Next let $v \in V$. Thus for some $v' \in V'$,

$$Tv = Tv'$$

so that $v - v' \in \ker T$.

We have $\ker T \subset V'$, otherwise we may add to $B$ a nonzero vector in $\ker T$, to obtain a Jordan basis of a subspace properly containing $V'$.

Thus $v \in V'$ so that $V' = V$, a contradiction.
Remarks: (1) The proof works for any algebraically closed field. (2) The above proof leads to the construction of a Jordan basis:

Example: $T$ is nilpotent of index $m$, i.e., $T^m = 0$ and $T^{m-1} \neq 0$.

1. Start with a basis of $T^{m-1}(V)$.
2. For each chosen element $v$ we pick an element $v'$ in $T^{m-2}(V)$ such that $T(v') = v$.
3. By adding elements in $\ker T \cap T^{m-2}(V)$ (thus start a new Jordan sequence) we obtain a Jordan basis of $T^{m-2}(V)$, etc.
THANK YOU FOR YOUR ATTENTION