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Jordan Normal Form Revisited

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Let us start with some online conversations.

conversation 1

conversation 2

1. Similarity

Given $A, B \in \mathbb{C}_{n\times n}$. A is said to be similar to B if there is a nonsingular matrix P such that $P^{-1}AP = B$, denoted by $B \sim A$.

Fact: Similarity is an equivalence relation

1. Reflexive ($A \sim A$): $A = IAI = I^{-1}AI$ for all $A \in \mathbb{C}_{n\times n}$.
2. Symmetric ($A \sim B$ implies $A \sim B$): $P^{-1}AP = B$ implies $PBP^{-1} = A$.
3. Transitive ($A \sim B$ and $B \sim C$ imply $C \sim A$) $P^{-1}AP = B$ and $Q^{-1}BQ = C$ imply $(PQ)^{-1}A(PQ) = C$.

As an equivalence relation, similarity partitions $\mathbb{C}_{n\times n}$ into equivalence classes $\rightarrow$ representative (normal form, canonical form)

2. Jordan Normal Form

Each $A \in \mathbb{C}_{n\times n}$ is similar to a block diagonal matrix

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

where each block J_i is a square matrix of the form (Jordan block)

$$J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \in \mathbb{C}_{n_i\times n_i}.$$

Remarks: The eigenvalues λ_i and λ_k may not be distinct for $i \neq k$.

Example:

$$A = \begin{bmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

$P^{-1}AP = J$ where

$$J = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \qquad P = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

Check: $AP = PJ$.

A little History:

In 1870 the Jordan canonical form appeared in Treatise on substitutions and algebraic equations by Camille Jordan (1838-1922). It appears in the context of a canonical form for linear substitutions over the finite field of order a prime.

The Jordan of Gauss-Jordan elimination is Wilhelm Jordan (1842 to 1899).

Jordan algebras are called after the German physicist and mathematician Pascual Jordan (1902 to 1980).

3. Numerical unstable

Consider

$$A = \begin{bmatrix} 1 & 1 \\ \epsilon & 1 \end{bmatrix}.$$

If $\epsilon = 0$, then the Jordan normal form is simply $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. However, for $\epsilon \neq 0$, the Jordan normal form is

$$\begin{bmatrix} 1+\sqrt{\epsilon} & 0 \\ 0 & 1-\sqrt{\epsilon} \end{bmatrix}.$$

So it is hard to develop a robust numerical algorithm for the Jordan normal form. For this reason, the Jordan normal form is usually avoided in numerical analysis.

Matlab command `[P, J] = jordan(A)`

4. Applications

Remark: Jordan normal form may be useless for numerical linear algebra, it has a valid place in applied linear algebra.

For recent development, see

Stefano Serra-Capizzano, Jordan canonical form of the Google matrix: a potential contribution to the PageRank computation, SIAM J. Matrix Anal. Appl. 27 (2005), 305–312.

(1) (a) Every $A \in \mathbb{C}_{n\times n}$ is similar to a complex symmetric matrix.

(b) Every $A \in \mathbb{C}_{n\times n}$ is a product of two complex symmetric matrices; one of which can be chosen nonsingular.

(2) $A^m \to 0$ if and only if the eigenvalue moduli of A are less than 1.

Reason: It suffices to consider $J = \begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} = \lambda I + N \in \mathbb{C}_{k\times k}$.

Since $N^m = 0$ for all $m \geq k$, we have

$$J^m = (\lambda I + N)^m = \sum_{i=0}^{m} \binom{m}{i} \lambda^i N^{m-i} = \sum_{i=m-k+1}^{m} \binom{m}{i} \lambda^i N^{m-i}$$

(a) Since the diagonal entries of J^m are λ^m, if $J^m \to 0$, then $\lambda^m \to 0$, i.e., $|\lambda| < 1$.

(b) Conversely, if $|\lambda| < 1$, then

$$\left|\binom{m}{m-j} \lambda^{m-j}\right| = \left|\frac{m(m-1)(m-2)\cdots(m-j+1)\lambda^m}{j!\lambda^j}\right| \leq \left|\frac{m^j \lambda^m}{j!\lambda^j}\right| \to 0$$

as $m \to \infty$ by l'Hopital's rule.

It finds application in population growth.

(3) A is similar to its transpose A^T:

By Jordan normal form $A = PJP^{-1}$. So $A \sim A^T$ amounts to $J \sim J^T$. Now

$$\begin{bmatrix} \lambda & & \\ 1 & \ddots & \\ & 1 & \lambda \end{bmatrix} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \begin{bmatrix} \lambda & 1 & \\ & \ddots & 1 \\ & & \lambda \end{bmatrix} \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}$$

(4) Solution to the a linear system of ODE:

$$x'(t) = Ax(t), \quad x(0) = x_0$$

$$x(t) = Pe^{tJ}P^{-1}x_0 = P \begin{bmatrix} e^{tJ_1} & & \\ & \ddots & \\ & & e^{tJ_p} \end{bmatrix} P^{-1}x_0,$$

where

$$e^{tJ_i} = e^{t\lambda_i} \begin{bmatrix} 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-1}}{(n-1)!} \\ & 1 & t & \cdots & \frac{t^{n-2}}{(n-2)!} \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

5. Proofs

1. Brualdi, Richard A., The Jordan canonical form: an old proof. Amer. Math. Monthly 94 (1987), no. 3, 257–267.
2. Cater, S., An elementary development of the Jordan canonical form. Amer. Math. Monthly 69 (1962), no. 5, 391–393.
3. Filippov, A. F., A short proof of the theorem on reduction of a matrix to Jordan form. Moscow Univ. Bul., 26 1971 no. 2, 18–19.
4. Fletcher, R.; Sorensen, D. C., An algorithmic derivation of the Jordan canonical form. Amer. Math. Monthly 90 (1983), no. 1, 12–16.
5. Galperin, A.; Waksman, Z., An elementary approach to Jordan theory. Amer. Math. Monthly 87 (1980), no. 9, 728–732.
6. Gohberg, Israel; Goldberg, Seymour, A simple proof of the Jordan decomposition theorem for matrices. Amer. Math. Monthly 103 (1996), no. 2, 157–159.
7. Väliaho, H., An elementary approach to the Jordan form of a matrix. Amer. Math. Monthly 93 (1986), no. 9, 711–714.
8. ....

6. Basis change

We switch to the lower triangular version of Jordan normal form: Each $A \in \mathbb{C}_{n\times n}$ is similar to a block diagonal matrix, i.e., for some nonsingular P

$$P^{-1}AP = J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_p \end{bmatrix}$$

where each block J_i is a square matrix of the form (Jordan block)

$$J_i = \begin{bmatrix} \lambda_i & & & \\ 1 & \lambda_i & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_i \end{bmatrix} \in \mathbb{C}_{n_i\times n_i}.$$

An interpretation: $P^{-1}AP$ is the matrix representation with respect to a new basis given by the columns of P, since $P^{-1}(\cdot)P$ means a change of basis.

Partition $P = [P_1 \,|\, \cdots \,|\, P_p]$ accordingly. Let $P_i = [v_1 \,|\, v_2 \,|\, \cdots \,|\, v_{n_i}] \in \mathbb{C}_{n \times n_i}$. Jordan form amounts to

$$AP_i = P_i J_i = P_i(\lambda_i I + N_i), \qquad N_i = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$$

From $AP_i = P_i(\lambda_i I + N_i)$ we have

$$[Av_1 \,|\, Av_2 \,|\, \cdots \,|\, Av_{n_i}] = \lambda_i [v_1 \,|\, v_2 \,|\, \cdots \,|\, v_{n_i}] + [v_2 \,|\, v_3 \,|\, \cdots \, v_{n_i} \,|\, 0],$$

i.e.

$$[(A - \lambda_i I)v_1 \,|\, (A - \lambda_i I)v_2 \,|\, \cdots \,|\, (A - \lambda_i I)v_{n_i}] = [v_2 \,|\, v_3 \,|\, \cdots \,|\, v_{n_i} \,|\, 0].$$

In other words, $v_1, v_2, \ldots, v_{n_i}$ are related: set $v := v_1$, $m := n_i$, $\lambda := \lambda_i$,

$$v_1 = v, v_2 = (A - \lambda I)v, \ldots, \ldots, v_m = (A - \lambda I)^{m-1} v \qquad (chain)$$

7. A Short proof

Roitman, Moshe, A short proof of the Jordan decomposition theorem. Linear and Multilinear Algebra 46 (1999), no. 3, 245–247.

In terms of the language of linear operator:

Theorem 7.1. Let $T : V \to V$ be a linear operator acting on a finite dimensional space over $\mathbb{C}$. Then V has a T-Jordan basis, that is, an ordered basis which consists of Jordan sequences: a (T, λ)-Jordan sequences, where λ is a scalar, is a sequence of vectors of the form

$$v, (T - \lambda I)v, \ldots, (T - \lambda I)^{m-1}v$$

for $v \in V$ and $m \geq 1$ such that $(T - \lambda I)^m v = 0$.

We will use contradiction:

(1) Assuming that the theorem is false, let $T : V \to V$ be a counterexample with $\dim V \geq 2$ minimal, thus $V \neq 0$.

(2) T has an eigenvalue μ in $\mathbb{C}$. Replacing T by $T - \mu I$ we may assume that $\mu = 0$. Thus

$$\dim T(V) < \dim V.$$

(3) By the minimality of $\dim V$, $T(V)$ has a T_0-Jordan basis, where T_0 is the restriction of T to $T(V)$, i.e.,

$$T_0 = T|_{T(V)} : T(V) \to T(V), \quad T_0(w) = T(w)$$

(4) Let V' be a subspace of maximal dimension among all subspaces of V

(a) invariant under T, and

(b) with a Jordan basis, B, containing a basis of $T(V)$.

Clearly $T(V) \subset V' \subset V$.

(5) Claim: $T(V) = T(V')$. Clearly $T(V') \subset T(V)$ since $V' \subset V$.
To prove $T(V) \subset T(V')$ we will show $B \cap T(V) \subset T(V')$.
Indeed if $w \in B \cap T(V)$, then w belongs to a (T, λ)-Jordan sequence

$$u, (T - \lambda I)u, \ldots, (T - \lambda I)^{m-1}u \quad \in B \subset V'.$$

Case 1: $\lambda = 0$. If $w \neq u$ (not the first one), then $w = T^k u \in T(V')$, $1 \leq k \leq m$. If $w = u$, pick any $w' \in V$ such that $T(w') = w$ since $w \in B \cap T(V) \subset T(V)$. If $w' \notin V'$, then we may add w' to the above sequences thus extending B to a Jordan basis of a subspace properly containing V', a contradiction. Hence $w' \in V'$, i.e., $w \in T(V')$.
Case 2: $\lambda \neq 0$. Observation: for $v \in V'$

$$v \in T(V') \quad \Leftrightarrow \quad (T - \lambda I)v = Tv - \lambda v \in T(V').$$

Since $w \in B \subset V'$ and $(T - \lambda I)^m w = 0 \in T(V')$ we see that

$$(T - \lambda I)^{m-1}w \in T(V'), \quad (T - \lambda I)^{m-2}w \in T(V'), \quad \cdots, w \in T(V').$$

Since B contains a basis of $T(V)$, we have $T(V) = T(V')$.

Next let $v \in V$. Thus for some $v' \in V'$,

$$Tv = Tv'$$

so that $v - v' \in \ker T$.

We have $\ker T \subset V'$, otherwise we may add to B a nonzero vector in $\ker T$, to obtain a Jordan basis of a subspace properly containing V'.

Thus $v \in V'$ so that $V' = V$, a contradiction.

Remarks: (1) The proof works for any algebraically closed field.

(2) The above proof leads to the construction of a Jordan basis:

Example: T is nilpotent of index m, i.e., $T^m = 0$ and $T^{m-1} \neq 0$.

(1) Start with a basis of $T^{m-1}(V)$.

(2) For each chosen element v we pick an element v' in $T^{m-2}(V)$ such that $T(v') = v$.

(3) By adding elements in $\ker T \cap T^{m-2}(V)$ (thus start a new Jordan sequence) we obtain a Jordan basis of $T^{m-2}(V)$, etc.

THANK YOU FOR YOUR ATTENTION