Asymptotic Behavior, Matrix Decomposition, Lie Groups

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Discuss some Matrix Decompositions and their Asymptotic Behaviors and their Lie Group extensions.

joint work with Huajun Huang
1. SVD

SVD:

\[ A = USV \]

where \( U, V \) unitary, \( S \) diagonal.

**Theorem 1.1.** (Gelfand, 1941) Let \( A \in \mathbb{C}_{n \times n} \).

\[
\lim_{m \to \infty} \|A^m\|^{1/m} = r(A)
\]

where \( r(A) \) is the spectral radius of \( A \) and \( \|A\| \) is the spectral norm of \( A \).

- obtained earlier by Beurling (1938).

Arrange:

\[ s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A), \quad |\lambda_1(A)| \geq |\lambda_2(A)| \geq \cdots \geq |\lambda_n(A)|. \]
Rewrite Beurling-Gelfand:

\[ \lim_{m \to \infty} \|A^m\|_2^{1/m} = r(A) \iff \lim_{m \to \infty} [s_1(A^m)]^{1/m} = |\lambda_1(A)| \]

Yamamoto (1967):

\[ \lim_{m \to \infty} [s_i(A^m)]^{1/m} = |\lambda_i(A)|, \quad i = 1, \ldots, n. \]

- a natural generalization of Beurling-Gelfand (finite dim. case).
- Loesener (1976) rediscovered Yamamoto
- Mathias (1990 another proof)
- Johnson and Nylen (1990 generalized singular values)
- Nylen and Rodman (1990 Banach algebra)
Lie groups

- $\mathfrak{g} =$ real semisimple Lie algebra with connected noncompact Lie group $G$.
- $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ a fixed (algebra) Cartan decomposition of $\mathfrak{g}$
- $K \subset G$ the connected subgroup with Lie algebra $\mathfrak{k}$.
- $\mathfrak{a} \subset \mathfrak{p}$ a maximal abelian subspace.
- Fix a closed Weyl chamber $a_+$ in $\mathfrak{a}$ and set

$$A_+ := \exp a_+, \quad A := \exp a$$

An example

Each $A \in \mathbb{C}_{n \times n}$ has Hermitian decomposition:

$$A = \frac{A - A^*}{2} + \frac{A + A^*}{2}.$$
(Group) Cartan decomposition

\[ G = KA_+K \]

- \( k_1, k_2 \in K \) not unique in \( g = k_1a_+(g)k_2 \), the element \( a_+(g) \in A_+ \) is unique.

**CMJD for real semisimple \( G \):**

- \( h \in G \) is **hyperbolic** if \( h = \exp(X) \) where \( X \in \mathfrak{g} \) is real semisimple, that is, \( \text{ad} \ X \in \text{End} \ (\mathfrak{g}) \) is diagonalizable over \( \mathbb{R} \).

- \( u \in G \) is **unipotent** if \( u = \exp(N) \) where \( N \in \mathfrak{g} \) is nilpotent, that is, \( \text{ad} \ N \in \text{End} \ (\mathfrak{g}) \) is nilpotent.

- \( e \in G \) is **elliptic** if \( \text{Ad}(e) \in \text{Aut} \ (\mathfrak{g}) \) is diagonalizable over \( \mathbb{C} \) with eigenvalues of modulus 1.

Each \( g \in G \) can be **uniquely** written as

\[ g = ehu, \]

where \( e, h, u \) commute.
Extension of Yamamoto:

Write $g = e(g)h(g)u(g)$

Fact: $h(g)$ is conjugate to $b(g) \in A_+$. 

**Theorem 1.2.** (Huang and Tam, 2006) Given $g \in G$, let $b(g) \in A_+$ be the unique element in $A_+$ conjugate to the hyperbolic part $h(g)$ of $g$. Then

$$\lim_{m \to \infty} [a_+(g^m)]^{1/m} = b(g).$$

2. QR decomposition

Recall QR decomposition

\[ A = QR \]

Set

\[ a(A) := \text{diag} (r_{11}, \ldots, r_{nn}) \]

where \( A \) is written in column form

\[ A = [A_1 | \cdots | A_n] \]

Geometric interpretation of \( a(A) \):

\( r_{ii} \) is the distance between \( A_i \) and the span of \( A_1, \ldots, A_{i-1} \), \( i = 2, \ldots, n \).
Theorem 2.1. (Huang and Tam 2006?) Given $A, X, B \in \mathbb{C}_{n \times n}$ nonsingular. Let $X = Y^{-1} J Y$ be the Jordan decomposition of $X$, where $J$ is the Jordan form of $X$,

$$\text{diag } J = \text{diag } (\lambda_1, \ldots, \lambda_n)$$

satisfying $|\lambda_1| \geq \cdots \geq |\lambda_n|$. Then

$$\lim_{m \to \infty} \alpha (AX^m B)^{1/m} = \text{diag } (|\lambda_{\omega(1)}|, \ldots, |\lambda_{\omega(n)}|),$$

where the permutation $\omega$ is uniquely determined by $Y B = L \omega U$:

$$\text{rank } \omega(i|j) = \text{rank } Y B(i|j), \quad 1 \leq i, j \leq n.$$
Numerical experiments:
Computing the discrepancy between

$$[a(A^m)]^{1/m} \text{ and } |\lambda|$$

of randomly generated $A \in \text{GL}_n(\mathbb{C})$.

The graph of

$$\|[a(A^m)]^{1/m} - \text{diag } (|\lambda_1|, \ldots, |\lambda_n|)\|_2$$

versus $m$ ($m = 1, \ldots, 100$).
If we consider
\[ |a_1(X^m)^{1/m} - |\lambda_1|| \]
for the above example, convergence occurs.
Iwasawa decomposition:

\[ G = KAN, \quad g = k(g)a(g)n(g) \]

Bruhat decomposition

\[ G = \bigcup_{s \in W} N^- m_s M A N \]

is a disjoint union. So for each \( g \in G \), there is a unique \( s \in W \) such that \( g \in N^- m_s M A N \).

**Theorem 2.2.** (Huang and Tam) Let \( v', v, g \in G \). Let \( g = ehu \) be the CMJD of \( g \). Let \( h = y^{-1}b(g)y \) for some \( y \in G \), and \( yv \in N^- m_s M A N \) in the Bruhat decomposition. Then

\[
\lim_{t \to \infty} [a(v' g^t v)]^{1/t} = s^{-1} \cdot b(g) = m_s^{-1} b(g) m_s,
\]

where the limit is independent of \( v' \) and the choice of \( y \). If \( b(g) \) is regular, that is, \( b(g) \) is in the interior of \( A_+ \), then \( s \) is uniquely determined by \( g \) and \( v \).
3. Cholesky decomposition

**Cholesky decomposition:** Given a positive definite $A \in \mathbb{C}_{n \times n}$,

$$A = R^* R$$

where $R$ is an upper triangular matrix with positive diagonal entries.

**Spectral theorem:** There is a unitary matrix $Y$ such that

$$Y A Y^{-1} = \Lambda$$

is diagonal with nonincreasing diagonal entries.

**Theorem 3.1.** Let $A \in \mathbb{C}_{n \times n}$ be positive definite. Let $A = Y^{-1} \Lambda Y$ where $Y \in U(n)$ and $\lambda_1 \geq \cdots \geq \lambda_n > 0$. Then

$$\lim_{i \to \infty} d(A^m)^{1/m} = \text{diag} (\lambda_{\omega(1)}, \ldots, \lambda_{\omega(n)})$$

where $\omega \in S_n$ is uniquely determined by $Y = L\omega U$. 
• $G$ = connected noncompact Lie group
• $P = \exp p$

\[ AN \rightarrow P \quad an \mapsto an^*(an) = na^2n^* \]

is diffeomorphism onto. So each $p \in P$ can be uniquely written as

\[ p = na^2n^*, \quad a \in A, \quad n \in N. \]

Set

\[ d(p) := a^2. \]

Moreover

\[ p = k^{-1}bk \]

for some $k \in K$ where $b \in A_+$ is uniquely determined by $p$.

**Theorem 3.2.** Let $p \in P$. Let $p = y^{-1}by$ where $y \in K$ and $b \in A_+$. Then

\[ \lim_{m \to \infty} d(p^m)^{1/m} = s^{-1} \cdot b \]

where $s \in W$ is uniquely determined by $y \in N^{-m_s}MAN$. 
4. QR iteration

Define a sequence \( \{A_m\}_{m \in \mathbb{N}} \) of matrices with

\[
A_1 := A
\]

and

\[
A_{j+1} := R_j Q_j \quad \text{if} \quad A_j = Q_j R_j, \quad j = 1, 2, \ldots
\]

\[
A_1 = Q_1 R_1, \quad A_2 = R_1 Q_1 = Q_2 R_2, \quad A_3 = R_2 Q_2, \ldots
\]

Notice that

\[
A_{m+1} = Q_m^{-1} A_m Q_m.
\]

Similarity \( \rightarrow \) the eigenvalues of \( A \) are fixed in the process.
**Theorem 4.1.** (Francis, 1961/62) Suppose that the moduli of the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $A \in \text{GL}_n(\mathbb{C})$ are distinct:

$$|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n| (> 0).$$

Let

$$A = Y^{-1} \text{diag} (\lambda_1, \ldots, \lambda_n) Y.$$

Assume

$$Y = L \omega U,$$

where $\omega$ is a permutation, $L$ is lower $\Delta$ and $U$ is unit upper $\Delta$. Then

1. the strictly lower triangular part of $A_m$ converges to zero

2. $\text{diag } A_m \rightarrow \text{diag} (\lambda_{\omega(1)}, \ldots, \lambda_{\omega(n)})$

- Kublanovskaja (1961) also obtained the result.
Iwasawa iteration

- $G =$ semisimple connected noncompact Lie group

Recall Iwasawa decomposition:

$$G = KAN, \quad g = k(g)a(g)n(g)$$

We want to study the behavior of the sequence $\{g_i\}_{i \in \mathbb{N}}$ where

$$
\begin{align*}
    g_1 &:= g \\
    g_i &:= a(g_{i-1}) n(g_{i-1}) k(g_{i-1}), \quad i = 2, 3, \ldots ,
\end{align*}
$$

The sequence $\{g_i\}_{i \in \mathbb{N}}$ is called the Iwasawa iteration of $g \in G$. 
Theorem 4.2. (Holmes, Huang and Tam) Let \( g \in G \) such that \( b(g) \in A_+ \) is regular. Let \( y \in G \) such that \( ygy^{-1} = cb \), where \( c := ye(g)y^{-1} \in M \) and \( b := b(g) = yh(g)y^{-1} \in A_+ \). Suppose

\[
y = n^{-m_s}m a n \in N^{-m_s}M A N
\]

is a Bruhat decomposition of \( y \), and

\[
y^{-1}m_s = k a n \in K A N
\]

is the Iwasawa decomposition of \( y^{-1}m_s \). Then

1. \( \lim_{i \to \infty} k(g_i) = c_* := (m_s m^{-1}c(m_s m)) \in M \).

2. \( \lim_{i \to \infty} a(g_i) = b_* := (m_s m^{-1}b(m_s m)) = s^{-1} \cdot b \).

3. \( \lim_{i \to \infty} \left( c_*^{i-1} n(g_i) c_*^{-i+1} \right) = \bar{a}(b_*^{-1}c_*^{-1} \bar{n}c_* b_*) \bar{n}^{-1} \bar{a}^{-1} = b_*^{-1}c_*^{-1}k^{-1}g k \).
5. \( \lambda \)-Aluthge iteration

Given \( 0 < \lambda < 1 \), the \( \lambda \)-Aluthge transform of \( A \in \mathbb{C}^{n \times n} \):

\[
\Delta_{\lambda}(A) := P^\lambda U P^{1-\lambda},
\]

where \( A = UP \) is the polar decomposition of \( A \), that is, \( U \) is unitary and \( P \) is positive semidefinite.

**Theorem 5.1.** (Antezana, Massey and Stojanoff, 2005) Let \( A \in \mathbb{C}^{n \times n} \) and \( 0 < \lambda < 1 \).

1. Any limit point of the \( \lambda \)-Aluthge sequence \( \{\Delta_{\lambda}^m(A)\}_{m \in \mathbb{N}} \) is normal, with eigenvalues \( \lambda_1(A), \ldots, \lambda_n(A) \).

2. \( \lim_{m \to \infty} \|\Delta_{\lambda}^m(A)\| = r(A) \).

- **Yamazaki (2003)** obtained \( \lim_{m \to \infty} \|\Delta_{\lambda}^m(A)\| = r(A) \) for Hilbert space operator when \( \lambda = 1/2 \).
Cartan decomposition:

\[ K \times P \rightarrow G, \quad (k, p) \mapsto kp \]

is a diffeomorphism. So

\[ g = kp, \quad k \in K, \ p \in P. \]

Given \( 0 < \lambda < 1 \), the \( \lambda \)-Aluthge transform of \( \Delta_\lambda : G \rightarrow G \) is defined as

\[ \Delta_\lambda(g) := p^\lambda kp^{1-\lambda}, \]

where \( p^\lambda := e^{\lambda x} \in P \) if \( p = e^x \) for some \( x \in p \).

**Theorem 5.2.** (Huang and Tam) Let \( g \in G \).

1. The limit points of the sequence \( \{\Delta^m_\lambda(g)\}_{m \in \mathbb{N}} \) are normal.

2. \( \lim_{m \to \infty} [a_+ (\Delta^m_\lambda(g))] = b(g) \).
THANK YOU FOR YOUR ATTENTION