Integr. equ. oper. theory 99 (9999), 1–14
0378-620X/99000-0, DOI 10.1007/s00020-003-0000

Integral Equations and Operator Theory

Some exponential inequalities for Semisimple Lie group

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Abstract. Let $|||\cdot|||$ be any give unitarily invariant norm. We obtain some exponential relations in the context of semisimple Lie group. On one hand they extend the inequalities (1) $|||e^A||| \le |||e^{\mathrm{Re}\, A}|||$ for all $A \in \mathbb{C}_{n\times n}$, where $\mathrm{Re}\, A$ denotes the Hermitian part of A, and (2) $|||e^{A+B}||| \le |||e^A e^B|||$, where A and B are $n \times n$ Hermitian matrices. On the other hand, the inequalities of Weyl, Ky Fan, Golden-Thompson, Lenard-Thompson, Cohen, and So-Thompson are recovered. Araki's relation on $(e^{A/2}e^Be^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$, where A, B are Hermitian and $r \in \mathbb{R}$, is extended.

1. Introduction

A norm $|||\cdot||| : \mathbb{C}_{n\times n} \to \mathbb{R}$ is said to be unitary invariant if $|||A||| = |||UAV|||$ for all $U, V \in U(n)$. It is known [3, Theorem IX.3.1, Theorem IX.3.7] that for any unitarily invariant norm $|||\cdot||| : \mathbb{C}_{n\times n} \to \mathbb{R}$,

$$|||e^A||| \le |||e^{\mathrm{Re}\, A}|||, \quad A \in \mathbb{C}_{n\times n}, \tag{1.1}$$

$$|||e^{A+B}||| \le |||e^A e^B|||, \quad A, B \in \mathbb{C}_{n\times n} \text{ are Hermitian}, \tag{1.2}$$

where $\mathrm{Re}\, A$ denotes the Hermitian part of $A \in \mathbb{C}_{n\times n}$. Inequality (1.2) is a generalization of the famous Golden-Thompson inequality [6, 21]

$$\mathrm{tr}\, e^{A+B} \le \mathrm{tr}\,(e^A e^B), \qquad A, B \text{ Hermitian}. \tag{1.3}$$

It is because that the Ky Fan n-norm, denoted by $\|\cdot\|_n$, is unitarily invariant, where $\|A\|_n$ is the sum of the singular values of $A \in \mathbb{C}_{n\times n}$. See [16, 22, 1, 2] for some generalizations of Golden-Thompson's inequality.

2000 Mathematics Subject Classification. Primary 15A45, 22E46; Secondary 15A42
Key Words: singular values, eigenvalue moduli, spectral radius, pre-order

A result in [3, Theorem IX.3.5] implies that for any irreducible representation π of the general linear group $\mathrm{GL}_n(\mathbb{C})$,

$$|\pi(e^{A+B})| \le |\pi(e^{\mathrm{Re}\,A}e^{\mathrm{Re}\,B})|, \quad A, B \in \mathbb{C}_{n\times n}, \tag{1.4}$$

where $|X|$ denotes the spectral radius of the linear map X.

The inequalities (1.1) and (1.2) compare two matrix exponentials using unitarily invariant norm. Apparently unitarily invariant norm plays no role in the inequality (1.4). But we will obtain Theorem 3.1 as unified extension of (1.1), (1.2) and (1.4).

After the preliminary materials are introduced in Section 2, Theorem 3.1 is obtained in the context of semisimple Lie group. It contains two sets of inequalities concerning a pre-order of Kostant [14]. To further demonstrated the importance of Theorem 3.1, in a sequence of remarks, we derive from Theorem 3.1 the inequalities of

1. Weyl [3]: the moduli of the eigenvalues of A are log majorized by the singular values of $A \in \mathbb{C}_{n\times n}$.
2. Ky Fan [3]: the real parts of the eigenvalues of A are majorized by the real singular values of $A \in \mathbb{C}_{n\times n}$.
3. Lenard-Thompson [16, 22]: $|||e^{A+B}||| \le |||e^{A/2}e^{B}e^{A/2}|||$, where $A, B \in \mathbb{C}_{n\times n}$ are Hermitian.
4. Cohen [4]: the eigenvalues of the positive definite part of e^A (with respect to the usual polar decomposition) are log majorized by the eigenvalues of $e^{\mathrm{Re}\,A}$, where $A \in \mathbb{C}_{n\times n}$.
5. So-Thompson [18]: the singular values of e^A are weakly log majorized by the exponentials of the singular values of $A \in \mathbb{C}_{n\times n}$.

In Section 4 we extend, in the context of semisimple Lie group, Araki's result [1] on the relation of $(e^{A/2}e^{B}e^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$, where $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, $r \ge 0$.

2. Preliminaries

We recall some basic notions, especially a pre-order of Kostant and some results in [14].

A matrix in $\mathrm{GL}_n(\mathbb{C})$ is called *elliptic* (respectively *hyperbolic*) if it is diagonalizable with norm 1 (respectively real positive) eigenvalues. It is called *unipotent* if all its eigenvalues are 1. The *complete multiplicative Jordan decomposition* of $g \in \mathrm{GL}_n(\mathbb{C})$ asserts that $g = ehu$ for $e, h, u \in \mathrm{GL}_n(\mathbb{C})$, where e is elliptic, h is hyperbolic, u is unipotent, and these three elements commute. The decomposition is obvious when g is in a Jordan canonical form with diagonal entries (i.e., eigenvalues) $z_1, \cdots, z_n$, in which

$$e = \mathrm{diag}\left(\frac{z_1}{|z_1|}, \cdots, \frac{z_n}{|z_n|}\right), \qquad h = \mathrm{diag}\,(|z_1|, \cdots, |z_n|),$$

and $u = h^{-1}e^{-1}g$ is a unit upper triangular matrix. The above decomposition can be extended to semisimple Lie groups.

Let $\mathfrak{g}$ be a real semisimple Lie algebra. Let G be any connected Lie group having $\mathfrak{g}$ as its Lie algebra. An element $X \in \mathfrak{g}$ is called *real semisimple* if $\operatorname{ad} X \in \operatorname{End} \mathfrak{g}$ is diagonalizable over $\mathbb{R}$ and is called *nilpotent* if $\operatorname{ad} X \in \operatorname{End} \mathfrak{g}$ is a nilpotent endomorphism. An element $g \in G$ is called *hyperbolic* if $g = \exp X$, where $X \in \mathfrak{g}$ is real semisimple and is called *unipotent* if $g = \exp X$, where $X \in \mathfrak{g}$ is nilpotent. An element $g \in G$ is *elliptic* if $\operatorname{Ad} g \in \operatorname{Aut} \mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition (CMJD) [14, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu,$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the analytic group of $\mathfrak{k}$ so that $\operatorname{Ad} K$ is a maximal compact subgroup of $\operatorname{Ad} G$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace in $\mathfrak{p}$. Then $A := \exp \mathfrak{a}$ is the analytic subgroup of $\mathfrak{a}$. Let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. The Weyl group operates naturally in $\mathfrak{a}$ and A and the isomorphism $\exp : \mathfrak{a} \to A$ is a W-isomorphism.

For each real semisimple $X \in \mathfrak{g}$, let

$$c(X) := \operatorname{Ad} G(X) \cap \mathfrak{a}$$

denote the set of all elements in $\mathfrak{a}$ which are conjugate to X (via the adjoint representation of G). For each hyperbolic $h \in G$, let

$$C(h) := \{ghg^{-1} : g \in G\} \cap A$$

denote the set of all elements in A which are conjugate to h. It turns out that $X \in \mathfrak{g}$ ($h \in G$, $e \in G$) is real semisimple (hyperbolic, elliptic) if and only if it is conjugate to an element in $\mathfrak{a}$ (A, K, respectively) [14, Proposition 2.3 and 2.4]. Thus $c(X)$ and $C(h)$ are single W-orbits in $\mathfrak{a}$ and A respectively. Moreover

$$C(\exp(X)) = \exp c(X).$$

Denote by $\operatorname{conv} W(X)$ the convex hull of the orbit $Wc(X) \subset \mathfrak{a}$ under the action of the Weyl group W. For arbitrary $g \in G$, define

$$C(g) := C(h(g)),$$

where $h(g)$ is the hyperbolic component of g and

$$\mathcal{A}(g) := \exp(\operatorname{conv} W(\log h(g)))$$

(For a hyperbolic $h \in G$, we write $\log h = X$ if $e^X = h$ and X is real semisimple. The element X is unique since $\operatorname{Ad}(e^X) = e^{\operatorname{ad} X}$ and the restriction of the usual matrix exponential map $e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$ on the set of diagonalizable matrices over $\mathbb{R}$ is one-to-one). Clearly $\mathcal{A}(g) \subset A$ and is invariant under the Weyl group. It is the "convex hull" of $C(g)$ in the multiplicative sense. Given $f, g \in G$, we say that $f \prec g$ if

$$\mathcal{A}(f) \subset \mathcal{A}(g),$$

or equivalently

$$C(f) \subset \mathcal{A}(g).$$

Notice that $\prec$ is a pre-order on G and $\mathcal{A}(\ell g \ell^{-1}) = \mathcal{A}(g)$ since $h(\ell g \ell^{-1}) = \ell h(g) \ell^{-1}$ for all $\ell \in G$. It induces a partial order on the equivalence classes of hyperbolic elements under the conjugation of G. The order $\prec$ is different from Thompson's pre-order [22] on $\mathrm{SL}_n(\mathbb{C})$ which simplifies the one made by Lenard [16]. Indeed the orders of Lenard and Thompson agree on the space of positive definite matrices.

We denote by $\hat{G}$ the index set of the irreducible representations of G, $\pi_\lambda : G \to \mathrm{Aut}\,(V_\lambda)$ a fixed representation in the class corresponding to $\lambda \in \hat{G}$, $|\pi_\lambda(g)|$ the spectral radius of the automorphism $\pi_\lambda(g) : V_\lambda \to V_\lambda$, where $g \in G$, that is, the maximum modulus of the eigenvalues of $\pi_\lambda(g)$, and χ_λ the character of π_λ. The following nice result of Kostant describes the pre-order $\prec$ via the irreducible representations of G and plays an important role in the coming sections.

Theorem 2.1. (Kostant [14, Theorem 3.1]) Let $f, g \in G$. Then $f \prec g$ if and only if $|\pi_\lambda(f)| \le |\pi_\lambda(g)|$ for all $\lambda \in \hat{G}$, where $|\cdot|$ denotes the spectral radius.

The following proposition describes $\prec$ in terms of inequalities when $G = \mathrm{SL}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

Proposition 2.2. Let $G = \mathrm{SL}_n(\mathbb{F})$, $\mathbb{F} = \mathbb{R}, \mathbb{C}$ and let $f, g \in G$. Denote by $\alpha_1, \ldots, \alpha_n$ the eigenvalues of f and $\beta_1, \ldots, \beta_n$ the eigenvalues of g arranged in the way that $|\alpha_1| \ge |\alpha_2| \ge \cdots \ge |\alpha_n|$ and $|\beta_1| \ge |\beta_2| \ge \cdots \ge |\beta_n|$. Then $f \prec g$ if and only if $|\alpha|$ is multiplicatively majorized by $|\beta|$, that is,

$$\begin{aligned} \prod_{i=1}^k |\alpha_i| &\le \prod_{i=1}^k |\beta_i|, \quad k = 1, \ldots, n-1, \\ \prod_{i=1}^n |\alpha_i| &= \prod_{i=1}^n |\beta_i|. \end{aligned}$$

Proof. We just deal with the real case (the complex case is similar) and we first describe the CMJD. Let $G = \mathrm{SL}_n(\mathbb{R})$ with $K = \mathrm{SO}(n)$, $A \subset \mathrm{SL}_n(\mathbb{R})$ consists of positive diagonal matrices of determinant 1, and $\mathfrak{a}$ is the space of diagonal matrices of zero trace. Now $\mathrm{Ad}\, g = g(\cdot)g^{-1}$, $g \in \mathrm{SL}_n(\mathbb{R})$, that is, $\mathrm{Ad}\, g$ is the conjugation via g. It is known that $s \in \mathfrak{sl}_n(\mathbb{R})$ real semisimple means that s is diagonalizable over $\mathbb{R}$ (see [11, Theorem 6.4] and [15, 558]); $n \in \mathfrak{sl}_n(\mathbb{R})$ nilpotent means $n^k = 0$ for some integer $k > 0$. So $h \in \mathrm{SL}_n(\mathbb{R})$ hyperbolic means that h is diagonalizable over $\mathbb{R}$ and the eigenvalues of h are positive; $e \in \mathrm{SL}_n(\mathbb{R})$ elliptic means that e is diagonalizable over $\mathbb{R}$ and the eigenvalues of e have modulus 1; $u \in \mathrm{SL}_n(\mathbb{R})$ is unipotent if $u - 1 \in \mathfrak{sl}_n(\mathbb{R})$ is nilpotent. Then follow [8, Lemma 7.1]: viewing

$g \in \mathrm{SL}_n(\mathbb{R})$ as an element in $\mathfrak{gl}_n(\mathbb{R})$, the additive Jordan decomposition [12, p.153] for $\mathfrak{gl}_n(\mathbb{R})$ yields

$$g = s + n_1$$

($s \in \mathrm{SL}_n(\mathbb{R})$ semisimple, that is, diagonalizable over $\mathbb{C}$, $n_1 \in \mathfrak{sl}_n(\mathbb{R})$ nilpotent and $sn_1 = n_1 s$). Moreover these conditions determine s and n_1 completely [11, Proposition 4.2]. Put $u := 1 + s^{-1}n_1 \in \mathrm{SL}_n(\mathbb{R})$ and we have the multiplicative Jordan decomposition

$$g = su,$$

where s is semisimple, u is unipotent, and $su = us$. By the uniqueness of the additive Jordan decomposition, s and u are also completely determined. Since s is diagonalizable,

$$s = eh,$$

where e is elliptic, h is hyperbolic, $eh = he$, and these conditions completely determine e and h. The decomposition can be obtained by observing that there is $k \in \mathrm{SL}_n(\mathbb{C})$ such that

$$k^{-1}sk = s_1 I_{r_1} \oplus \cdots \oplus s_m I_{r_m},$$

where $s_1 = e^{i\xi_1}|s_1|, \ldots, s_m = e^{i\xi_m}|s_m|$ are the distinct eigenvalues of s with multiplicities $r_1, \ldots, r_m$ respectively. Set

$$e := k(e^{i\xi_1} I_{r_1} \oplus \cdots \oplus e^{i\xi_m} I_{r_m})k^{-1}, \quad h := k(|s_1| I_{r_1} \oplus \cdots \oplus |s_m| I_{r_m})k^{-1}.$$

Since

$$ehu = g = ugu^{-1} = ueu^{-1}uhu^{-1}u,$$

the uniqueness of s, u, e and h implies e, u and h commute. Since g is fixed under complex conjugation, the uniqueness of e, h and u imply $e, h, u \in \mathrm{SL}_n(\mathbb{R})$ [8, p.431]. Thus $g = ehu$ is the CMJD for $\mathrm{SL}_n(\mathbb{R})$. The eigenvalues of h are simply the eigenvalue moduli of s and thus of g.

We now are to describe $\prec$. Let $\mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n) + \mathfrak{p}$ be the fixed Cartan decomposition of $\mathfrak{sl}_n(\mathbb{R})$, that is, $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ is the space of real symmetric matrices of zero trace. So $K = \mathrm{SO}(n)$. Let $\mathfrak{a} \subset \mathfrak{p}$ be the maximal abelian subspace of $\mathfrak{sl}_n(\mathbb{R})$ in $\mathfrak{p}$ containing the diagonal matrices. So the analytic group A of $\mathfrak{a}$ is the group of positive diagonal matrices of determinant 1. The Weyl group W of $(\mathfrak{a}, \mathfrak{g})$ is the full symmetric group S_n [13] which acts on A and $\mathfrak{a}$ by permuting the diagonal entries of the matrices in A and $\mathfrak{a}$. Now

$$C(f) := C(h(f)) = \{\mathrm{diag}\,(|\alpha_{\sigma(1)}|, \cdots, |\alpha_{\sigma(n)}|) : \sigma \in S_n\},$$

where $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of $f \in \mathrm{SL}_n(\mathbb{C})$ with the order $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$. So

$$c(\log h(f)) = \{\mathrm{diag}\,(\log|\alpha_{\sigma(1)}|, \cdots, \log|\alpha_{\sigma(n)}|) : \sigma \in S_n\}$$

and

$$\mathcal{A}(f) = \exp \mathrm{conv}\,\{\mathrm{diag}\,(\log|\alpha_{\sigma(1)}|, \cdots, \log|\alpha_{\sigma(n)}|) : \sigma \in S_n\}.$$

So $f \prec g$, $f, g \in \mathrm{SL}_n(\mathbb{R})$ means that $\log|\alpha|$ is majorized by $\log|\beta|$ [3, p.33], usually denoted by $|\alpha| \prec_{\log} |\beta|$ and is called log majorization [2], where β's are the eigenvalues of g. □

Remark 2.3. In the above example, the pre-order $\prec$ in $\mathrm{SL}_n(\mathbb{R}) \subset \mathrm{SL}_n(\mathbb{C})$ coincides with that in $\mathrm{SL}_n(\mathbb{C})$ since the Weyl groups are identical. But it is pointed out in [14, Remark 3.1.1] that the pre-order $\prec$ is not necessarily the same as the pre-order on the semisimple G that would be induced by a possible embedding of G in $\mathrm{SL}_n(\mathbb{C})$ for some n.

3. A pre-order of Kostant and some order relations

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the real semisimple Lie algebra $\mathfrak{g}$. For each $X \in \mathfrak{g}$, write $X = X_{\mathfrak{k}} + X_{\mathfrak{p}}$, where $X_{\mathfrak{k}} \in \mathfrak{k}$ and $X_{\mathfrak{p}} \in \mathfrak{p}$. Let

$$G = KP$$

be the Cartan decomposition of analytic group G of $\mathfrak{g}$ [8], where $P := \exp \mathfrak{p}$.

Define $g^* := pk^{-1}$ if $g = kp$ with respect to the Cartan decomposition $G = KP$. When $G = \mathrm{SL}_n(\mathbb{C})$ with $K = \mathrm{SU}(n)$, g^* is simply the complex conjugate transpose of g.

Theorem 3.1. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Then for any $g \in G$,

$$g^{2n} \prec (g^*)^n g^n \prec (g^* g)^n, \qquad n = 1, 2, \ldots. \tag{3.1}$$

Moreover for any $X, Y \in \mathfrak{g}$,

$$e^{X+Y} \prec e^{-\theta(X+Y)/2} e^{(X+Y)/2} \prec e^{X_{\mathfrak{p}}} e^{Y_{\mathfrak{p}}}, \tag{3.2}$$

where θ is the Cartan involution of $\mathfrak{g}$ with respect to the given Cartan decomposition.

Remark 3.2. When $G = \mathrm{SL}_n(\mathbb{C})$ or $\mathrm{GL}_n(\mathbb{C})$, the relation ${g^*}^n g^n \prec (g^*g)^n$ was established in [4] and $g^{2n} \prec (g^*g)^n$ was obtained in [22]. The inequality $g^{2n} \prec (g^*)^n g^n$ is reduced to Weyl's inequality by Proposition 2.2. See Remark 3.8. Kostant [14, proof of Theorem 6.3] also proved $g^{2n} \prec (g^*g)^n$ and $e^{A+B} \prec e^A e^B$, $A, B \in \mathfrak{p}$, for general G. The generalization as a whole is new.

Proof. Let $\theta \in \mathrm{Aut}\, \mathfrak{g}$ be the Cartan involution of $\mathfrak{g}$, that is, θ is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. Set $P = e^{\mathfrak{p}}$. We have the (global) Cartan decomposition

$$G = KP.$$

The involution θ induces an automorphism Θ of G such that the differential of Θ at the identity is θ [13, p.387]. Explicitly

$$\Theta(kp) = kp^{-1}, \quad k \in K, \ p \in P.$$

For any $g \in G$ let

$$g^* := \Theta(g^{-1}).$$

If $g = kp$, then
$$g^* = \Theta(p^{-1}k^{-1}) = \Theta(p^{-1})k^{-1} = pk^{-1},$$
and hence $g^*g = p^2 \in P$, since the centralizer $G^\Theta = \{g \in G : \Theta(g) = g\}$ coincides with K [13, p.305]. So
$$g^* := \Theta(g^{-1}) = (\Theta(g))^{-1}, \quad (g^*)^* = g, \quad (fg)^* = g^*f^*, \quad (g^*)^n = (g^n)^*,$$
for all $f, g \in G$, $n \geq 1$. Since θ is the differential of Θ at the identity, we have [8, 110]
$$\Theta(e^X) = e^{\theta X},$$
for all $X \in \mathfrak{g}$. So
$$(e^X)^* = \Theta(e^{-X}) = e^{-\theta X}. \tag{3.3}$$

The relation $g^{2n} \prec (g^*g)^n$ in (3.1) is known in [14, p.448] and we use similar idea to establish (3.1). Actually the original idea can be found in [22] when $G = \mathrm{SL}_n(\mathbb{C})$.

We denote by $\Pi_\lambda : \mathfrak{g} \to \mathrm{End}\, V_\lambda$ the differential at the identity of the representation $\pi_\lambda : G \to \mathrm{Aut}\, V_\lambda$. So [8, p.110]
$$\exp \circ \Pi_\lambda = \pi_\lambda \circ \exp, \tag{3.4}$$
where the exponential function on the left is $\exp : \mathrm{End}\, V_\lambda \to \mathrm{Aut}\, V_\lambda$ and the one on the right is $\exp : \mathfrak{g} \to G$. Now $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ (direct sum) is a compact real form of $\mathfrak{g}_\mathbb{C}$ (the complexification of $\mathfrak{g}$). The representation $\Pi_\lambda : \mathfrak{g} \to \mathrm{End}\, V_\lambda$ naturally defines a representation $\mathfrak{u} \to \mathrm{End}\, V_\lambda$ of $\mathfrak{u}$, also denoted by Π_λ and vice versa. Let U be a simply connected Lie group of $\mathfrak{u}$ [24, p.101] so that it is compact [5, Corollary 3.6.3]. There is a unique homomorphism $\hat{\pi}_\lambda : U \to \mathrm{Aut}\, V_\lambda$ such that the differential of $\hat{\pi}_\lambda$ at the identity is Π_λ [24, Theorem 3.27]. Thus there exists an inner product $\langle \cdot, \cdot \rangle$ on V_λ such that $\hat{\pi}_\lambda(u)$ is orthogonal for all $u \in U$. We will assume that V_λ is endowed with this structure from now on. Differentiate the identity
$$\langle \hat{\pi}_\lambda(e^{tZ})X, \hat{\pi}_\lambda(e^{tZ})Y \rangle = \langle X, Y \rangle,$$
for all $X, Y \in V_\lambda$ at $t = 0$ we have
$$\langle \Pi_\lambda(Z)X, Y \rangle = -\langle X, \Pi_\lambda(Z)Y \rangle$$
by (3.4). Thus, with respect to $\langle \cdot, \cdot \rangle$, $\Pi_\lambda(Z)$ is skew Hermitian for all $Z \in \mathfrak{u}$ [13, Proposition 4.6], [14, p.435]. Then $\Pi_\lambda(Z)$ is skew Hermitian if $Z \in \mathfrak{k}$ and is Hermitian if $Z \in \mathfrak{p}$. So $\pi_\lambda(z)$ is unitary if $z \in K$ and is positive definite if $z \in P$ by (3.4).

Since each $g \in G$ can be written as $g = kp$, $k \in K$ and $p \in P$,
$$\begin{aligned}
\langle u, \pi_\lambda(g^*)v \rangle &= \langle u, \pi_\lambda(pk^{-1})v \rangle \\
&= \langle u, \pi_\lambda(p)\pi_\lambda(k^{-1})v \rangle \\
&= \langle \pi_\lambda(k)\pi_\lambda(p)u, v \rangle \\
&= \langle \pi_\lambda(g)u, v \rangle,
\end{aligned}$$
for all $u, v \in V_\lambda$. Thus
$$\pi_\lambda(g)^* = \pi_\lambda(g^*), \tag{3.5}$$

where $\pi_\lambda(g)^*$ denotes the (Hermitian) adjoint of $\pi_\lambda(g)$. Thus $\pi_\lambda(g^*g) = \pi_\lambda(g)^*\pi_\lambda(g) \in \operatorname{Aut} V_\lambda$ is a positive definite operator for all $g \in G$. Denote by

$$\|\pi_\lambda(g)\| := \max_{0\neq v\in V_\lambda} \frac{\|\pi_\lambda(g)v\|}{\|v\|},$$

the operator norm of $\pi_\lambda(g) \in \operatorname{Aut} V_\lambda$, where $\|v\| := \langle v, v\rangle^{1/2}$ is the norm induced by $\langle\cdot,\cdot\rangle$. Thus the spectral theorem for self-adjoint operators implies

$$|\pi_\lambda(p)| = \|\pi_\lambda(p)\|, \quad \text{for all } p \in P.$$

Because of Theorem 2.1, to arrive at the claim (3.1) it suffices to show

$$|\pi_\lambda(g^{2n})| \le |\pi_\lambda((g^*)^n g^n)| \le |\pi_\lambda((g^*g)^n)|, \quad \text{for all } \lambda \in \hat{G}.$$

Now

$$\begin{aligned} |\pi_\lambda((g^*)^n g^n)| &= |\pi_\lambda((g^n)^* g^n)| \\ &= \|\pi_\lambda((g^n)^* g^n)\| \qquad \text{since } \pi_\lambda((g^n)^* g^n) \in \operatorname{End} V_\lambda \text{ is p.d.} \\ &= \|\pi_\lambda(g^n)^* \pi_\lambda(g^n)\| \qquad \text{by (3.5)} \\ &= \|\pi_\lambda(g^n)\|^2 \qquad \text{since } \|T\|^2 = \|T^*T\|. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\pi_\lambda((g^*g)^n)| &= |\pi_\lambda(g^*g)|^n \\ &= \|\pi_\lambda(g^*g)\|^n \qquad \text{since } \pi_\lambda((g^*g) \in \operatorname{End} V_\lambda \text{ is p.d.} \\ &= \|\pi_\lambda(g)^*\pi_\lambda(g)\|^n \\ &= \|\pi_\lambda(g)\|^{2n} \qquad \text{since } \|T\|^2 = \|T^*T\| \\ &\ge \|\pi_\lambda(g^n)\|^2 \qquad \text{since } \|T^n\| \le \|T\|^n, \end{aligned}$$

where the inequality is due to the well known fact that the spectral radius is no greater than the operator norm. So we have $(g^*)^n g^n \prec (g^*g)^n$. Now

$$|\pi_\lambda((g^*)^n)g^n)| = |\pi_\lambda((g^n)^*)\pi_\lambda(g^n)| = \|\pi_\lambda(g^n)\|^2 \ge |\pi_\lambda(g^n)|^2 = |\pi_\lambda(g^{2n})|.$$

Hence $g^{2n} \prec (g^*)^n g^n$ and we just proved the claim.

By the first relation in (3.1), if $g = xy$, where $x, y \in G$, then for any $m \in \mathbb{N}$,

$$(xy)^{2^{m+1}} \prec (y^*x^*)^{2^m}(xy)^{2^m}.$$

Set $x = e^{X/2^m}$, $y = e^{Y/2^m}$, where $X, Y \in \mathfrak{g}$. From (3.3)

$$\begin{aligned} ((e^{X/2^m}e^{Y/2^m})^{2^m})^2 &\prec ((e^{Y/2^m})^*(e^{X/2^m})^*)^{2^m}(e^{X/2^m}e^{Y/2^m})^{2^m} \\ &= (e^{-\theta Y/2^m}e^{-\theta X/2^m})^{2^m}(e^{X/2^m}e^{Y/2^m})^{2^m}. \end{aligned}$$

Since $\lim_{t\to\infty}(e^{X/t}e^{Y/t})^t = e^{X+Y}$ [8, p.115] (Lie-Trotter formula; as pointed out in [7, p.35], Trotter's formula is for suitable unbounded operators on an infinite dimensional Hilbert space [17, VIII.8]), and the relation $\prec$ remains valid as we take limits on both sides because the spectral radius is a continuous function on $\operatorname{Aut} V_\lambda$, by Theorem 2.1 we have $e^{2(X+Y)} \prec e^{-\theta(X+Y)}e^{(X+Y)}$. As a result

$$e^{X+Y} \prec e^{-\frac{1}{2}\theta(X+Y)}e^{\frac{1}{2}(X+Y)}, \quad \text{for all } X, Y \in \mathfrak{g}$$

and we just established the first part of (3.2).

Let $g = e^{(X+Y)/n}$, $X, Y \in \mathfrak{g}$. By the second relation of (3.1) and (3.3)

$$(e^{-\theta(X+Y)/n})^n (e^{(X+Y)/n})^n \prec ((e^{-\theta(X+Y)/n} e^{(X+Y)/n}))^n.$$

As before

$$e^{-\theta(X+Y)} e^{X+Y} \prec e^{2(X+Y)_{\mathfrak{p}}} = e^{2X_{\mathfrak{p}}+2Y_{\mathfrak{p}}} \prec e^{2X_{\mathfrak{p}}} e^{2Y_{\mathfrak{p}}},$$

where the last relation is established in [14, Theorem 6.3]. □

Similar technique of the proof is also used in [19, 20]. By setting $Y = 0$ or $Y = X$ in the second set of inequalities of Theorem 3.1, we have

Corollary 3.3. Let $X \in \mathfrak{g}$. Then $e^X \prec e^{-\theta X/2} e^{X/2} \prec e^{X_{\mathfrak{p}}}$.

Remark 3.4. The statement $e^{X+Y} \prec e^{X_{\mathfrak{k}}} e^{Y_{\mathfrak{k}}}$ is not true by simply considering $G = \mathrm{SL}_n(\mathbb{C})$ in which $K = \mathrm{SU}(n)$ and $\mathfrak{k} = \mathfrak{su}(n)$. Clearly $e^{X_{\mathfrak{k}}} e^{Y_{\mathfrak{k}}} \in \mathrm{SU}(n)$ and we may pick $X, Y \in \mathfrak{sl}_n(\mathbb{C})$ such that $X + Y$ is nonzero Hermitian matrix with a positive eigenvalue. Viewing each $g \in \mathrm{SL}_n(\mathbb{C})$ as a linear operator on $V_\lambda = \mathbb{C}^n$ (the natural representation of $\mathrm{SL}_n(\mathbb{C})$), the spectral radius $|e^{X_{\mathfrak{k}}} e^{Y_{\mathfrak{k}}}| = 1$ but $|e^{X+Y}| > 1$.

Remark 3.5. (Cohen's inequalities) When $G = \mathrm{GL}_n(\mathbb{C})$ the second relation in (3.1)

$$g^{*n} g^n \prec (g^* g)^n, \qquad n = 1, 2, \ldots$$

is equivalent to

$$p(g^n) \prec (p(g))^n, \qquad n = 1, 2, \ldots$$

where $g = k(g)p(g)$ is the polar decomposition of $g \in G$. If we set $g = e^{X/n}$, then

$$p(e^X) \prec [p(e^{X/n})]^n, \qquad n = 1, 2, \ldots$$

Now $p(e^{X/n}) = ((e^{X/n})^* e^{X/n})^{1/2} = (e^{-\theta X/n} e^{X/n})^{1/2}$. By $\lim_{t\to\infty} (e^{X/t} e^{Y/t})^t = e^{X+Y}$, we have

$$\lim_{n\to\infty} [p(e^{X/n})]^n = \lim_{n\to\infty} [(e^{-\theta X/n} e^{X/n})^n]^{1/2} = [\lim_{n\to\infty} (e^{-\theta X/n} e^{X/n})^n]^{1/2} = e^{X_{\mathfrak{p}}},$$

and thus

$$p(e^X) \prec e^{X_{\mathfrak{p}}}.$$

In particular the singular values of e^A is log majorized by the eigenvalue moduli of $e^{\mathrm{Re}\, A}$, i.e., Cohen's result [4] when $G = \mathrm{GL}_n(\mathbb{C})$ (with appropriate scaling on $\mathrm{SL}_n(\mathbb{C})$).

Remark 3.6. (Ky Fan's inequality and inequality (1.1))

Continuing with Proposition 2.2, for $A \in \mathfrak{sl}_n(\mathbb{C})$, the moduli of the eigenvalues of e^A are the exponentials of the real parts of the eigenvalues of A, counting multiplicities. The matrix $e^{\mathrm{Re}\, A}$ is positive definite. So the eigenvalues of $e^{\mathrm{Re}\, A}$ are indeed the singular values, and are the exponentials of the eigenvalues of $\mathrm{Re}\, A$. The eigenvalues of $\mathrm{Re}\, A$ are known as the real singular values of A, denoted by

$\beta_1 \geq \cdots \geq \beta_n$. Denote the real parts of the eigenvalues of A by $\alpha_1 \geq \cdots \geq \alpha_n$. By Corollary 3.3 $e^A \prec e^{\mathrm{Re}\, A}$ which amounts to

$$\begin{aligned} \prod_{i=1}^k e^{\alpha_i} &\leq \prod_{i=1}^k e^{\beta_i}, \quad i = 1, \ldots, n-1, \\ \prod_{i=1}^n e^{\alpha_i} &= \prod_{i=1}^n e^{\beta_i}, \end{aligned}$$

that is $e^\alpha \prec_{\log} e^\beta$. Thus, by taking log on the above relation, the relation $e^A \prec e^{\mathrm{Re}\, A}$ amounts to the usual majorization relation $\alpha \in \mathrm{conv}\, S_n \beta$, a well known result of Ky Fan [3, Proposition III.5.3] for $\mathfrak{gl}_n(\mathbb{C})$ with appropriate scaling on $\mathfrak{sl}_n(\mathbb{C})$).

From the second relation of Corollary 3.3, $e^A e^{A^*} \prec e^{A+A^*}$ which amounts to the fact that the singular values of e^A (that is, the square roots of the eigenvalues of $e^A e^{A^*}$) are multipicatively majorized, and hence weakly majorized [3, p.42], [2], by the singular values (also the eigenvalues) of the positive definite $e^{\mathrm{Re}\, A}$. Thus

$$|||e^A||| \leq |||e^{\mathrm{Re}\, A}|||,$$

for all unitarily invariant norms $|||\cdot|||$ [3, Theorem IX.3.1] by Ky Fan Dominance Theorem [3, Theorem IV.2.2]. Thus we have (1.1).

Remark 3.7. (So-Thompson's inequality)

For $A \in \mathbb{C}_{n\times n}$, So-Thompson inequalities [18, Theorem 2.1] asserts that

$$\prod_{i=1}^k s_i(e^A) \leq \prod_{i=1}^k e^{s_i(A)}, \quad k = 1, \ldots, n.$$

From $e^A e^{A^*} \prec e^{A+A^*}$, $A \in \mathbb{C}_{n\times n}$, So-Thompson inequalities can be derived via Fan-Hoffman inequalities [3, proposition III.5.1]

$$\lambda_i(\mathrm{Re}\, A) \leq s_i(A), \quad i = 1, \ldots, n,$$

where $s_1(A) \geq \cdots \geq s_n(A)$ denote the singular values of $A \in \mathbb{C}_{n\times n}$.

Remark 3.8. (Weyl's inequality and inequalities (1.2) and (1.4))

Let $A \in \mathrm{SL}_n(\mathbb{C})$. By (3.5) $A^2 \prec A^*A$. By Proposition 2.2, $|\lambda^2(A)| \prec_{\log} |\lambda(A^*A)| = |s(A^*A)|$, that is,

$$|\lambda(A)| \prec_{\log} s(A).$$

By scaling and continuity argument, the log majorization remains valid for $A \in \mathbb{C}_{n\times n}$, that is, Weyl's inequality [3, p.43]. In the literature, Weyl's inequality is often proved via the kth exterior power once $|\lambda_1(A)| \leq s_1(A)$ is established, for example [3, p.42-43]. Such an approach shares some favor of Theorem 2.1.

If $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, then e^A, e^B and e^{A+B} are positive definite. Though $e^A e^B$ is not positive definite in general, its eigenvalues, denoted by $\delta_1 \geq \cdots \geq \delta_n$, are positive since $e^A e^B$ and the positive definite $e^{A/2} e^B e^{A/2}$ share the same eigenvalues, counting multiplicities. Denote the eigenvalues of e^{A+B} by $\gamma_1 \geq \cdots \geq \gamma_n$. Thus γ is multiplicatively majorized by δ because of $e^{A+B} \prec e^A e^B$ (Theorem 3.1). Notice that δ is also multiplicatively majorized by the singular values $s_1 \geq \cdots \geq s_n$ of $e^A e^B$, by Weyl's inequality. Hence we have the weak majorization relation $\gamma \prec_w s$ [3, p.42] so that (1.2) follows. Finally (1.4) follows from Theorem 3.1 and Theorem 2.1.

Remark 3.9. (Lenard-Thompson's inequality) Lenard's result [16] together with [22, Theorem 2] imply that

$$|||e^{A+B}||| \leq |||e^{A/2}e^B e^{A/2}|||, \quad A, B \in \mathbb{C}_{n\times n} \text{ Hermitian}, \tag{3.6}$$

from which Golden-Thompson's result follows. It is because e^{A+B} and $e^{A/2}e^B e^{A/2}$ are positive definite and their traces are indeed the Ky Fan n-norm, that is, sum of singular values which is unitarily invariant. Indeed Lenard's original result asserts that any arbitrary neighborhood of e^{A+B} contains X such that $X \prec e^{A/2}e^B e^{A/2}$ [16, p.458]. By a limit argument and Thompson's argument, (3.6) follows. The inequality (3.6) follows from the stronger relation:

$$e^{A+B} \prec e^{A/2}e^B e^{A/2}, \quad A, B \text{ Hermitian}. \tag{3.7}$$

Let us establish (3.7). From Theorem 3.1

$$e^{A+B} \prec e^A e^B, \quad A, B \text{ Hermitian}$$

is a generalization of Golden-Thompson's inequality (1.3). Now (3.7) is true because $\pi_\lambda(e^A e^B)$ and $\pi_\lambda(e^{A/2}e^B e^{A/2})$ have the same spectrum (by the fact that XY and YX have the same spectrum and π_λ is a representation) and thus have the same spectral radius. Then apply Theorem 2.1.

4. Extension of Araki's result

Araki's result [1] asserts that if $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, then

$$(e^{A/2}e^B e^{A/2})^r \prec e^{rA/2}e^{rB}e^{rA/2}, \qquad r > 1. \tag{4.1}$$

It appears in the proof of the main result in [1, p.168-169]. Also see [10] for a short proof. Notice that $e^{A/2}e^B e^{A/2}$ and $e^{rA/2}e^{rB}e^{rA/2}$ in (4.1) are positive definite so that their eigenvalues and singular values coincide. So (4.1) amounts to

$$s((e^{A/2}e^B e^{A/2})^r) \prec_{\log} s(e^{rA/2}e^{rB}e^{rA/2}), \qquad r > 1,$$

or equivalently

$$s((e^{qA/2}e^{qB}e^{qA/2})^{1/q}) \prec_{\log} s((e^{pA/2}e^{pB}e^{pA/2})^{1/p}), \qquad 0 < q \leq p.$$

Using (4.1) and Lie's product formula [8, Lemma 1.8, p.106]

$$e^{A+B} = \lim_{r\to 0}(e^{rA/2}e^{rB}e^{rA/2})^{1/r},$$

Golden-Thompson's result is strengthened [2]:

$$|||e^{pA/2}e^{pB}e^{pA/2}|||$$

decreases down to $|||e^{A+B}|||$ as $p \downarrow 0$ for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$ and in particular

$$\operatorname{tr} e^{A+B} \le \operatorname{tr}\,[e^{pA/2}e^{pB}e^{pA/2}]^{1/p}, \qquad p > 0.$$

Araki's result also implies a result of Wang and Gong [23] (also see [3, Theorem IX.2.9]).

In order to extend (4.1) for general G, we need a result of Heinz [9] concerning two positive semidefinite operators. Indeed the original proof of Araki's result [1] also makes use of Heinz's result. Given two positive semidefinite operators A, B, the spectrum (counting multiplicities) $\lambda(AB) = \lambda(A^{1/2}BA^{1/2})$ and thus all eigenvalues of AB are positive. So the largest eigenvalue of AB, $\lambda_1(AB)$, is the spectral radius of AB. The first part of the following theorem is due to Heinz [9] (see [p.255-256] for two nice proofs of Heinz's result). The second part is proved via the Heinz's result in [3, Theorem IX.2.6] in a somewhat lengthy way. See [19] for some generalization of Heniz's theorem.

Theorem 4.1. The following two statements are equivalent and valid.

1. (Heinz) For any two positive semidefinite operators A, B,
$$\|A^sB^s\| \le \|AB\|^s, \quad 0 \le s \le 1.$$
2. For any two positive semidefinite operators A, B,
$$\lambda_1(A^sB^s) \le \lambda_1^s(AB), \quad 0 \le s \le 1.$$

Proof. We just establish the equivalence of the two statements. Since $\|T\| = \|T^*T\|^2$,

$$\|A^sB^s\| = \|(A^sB^s)A^sB^s\|^{1/2} = \|B^sA^{2s}B^s\|^{1/2} = \lambda_1^{1/2}(B^sA^{2s}B^s) = \lambda_1^{1/2}(A^{2s}B^{2s}),$$

and

$$\|AB\|^s = \|ABBA\|^{s/2} = \lambda_1^{s/2}(AB^2A) = \lambda_1^{s/2}(A^2B^2).$$

□

Remark 4.2. An equivalent statement to Heniz's result is that for any positive operators A, B, $\|A^tB^t\| \ge \|AB\|^t$ if $t \ge 1$, or equivalently $\lambda_1(A^tB^t) \ge \lambda_1^t(AB)$ [3, p.256-257].

Since $P := e^{\mathfrak{p}}$, each element of P is of the form e^A, $A \in \mathfrak{p}$ so that $(e^A)^r := e^{rA} \in P$, where $r \in \mathbb{R}$. So $f^r, g^r \in P$, f^rg^r (hyperbolic, since f^rg^r is conjugate to $f^{r/2}g^rf^{r/2}$), $r \in \mathbb{R}$, are well defined for $f, g \in P$.

When $A, B \in \mathfrak{p}$, $e^{A/2}e^Be^{A/2} \in P$ since it is of the form g^*g, where $g = e^{B/2}e^{A/2}$. Thus $(e^{A/2}e^Be^{A/2})^r \in P$ $(r \in \mathbb{R})$ is well defined.

Theorem 4.3. Let $A, B \in \mathfrak{p}$. Then

$$\begin{aligned} (e^{A/2}e^Be^{A/2})^r &\prec e^{rA/2}e^{rB}e^{rA/2}, & r > 1,\\ e^{rA/2}e^{rB}e^{rA/2} &\prec (e^{A/2}e^Be^{A/2})^r, & 0 \le r \le 1. \end{aligned}$$

Moreover, for all $\lambda \in \hat{G}$

$$\begin{aligned} \chi_\lambda((e^{A/2}e^{B}e^{A/2})^r) &\le \chi_\lambda(e^{rA/2}e^{rB}e^{rA/2}), & r > 1, \\ \chi_\lambda(e^{rA/2}e^{rB}e^{rA/2}) &\le \chi_\lambda((e^{A/2}e^{B}e^{A/2})^r), & 0 \le r \le 1. \end{aligned}$$

Proof. Notice that $\pi_\lambda(e^A)$ is positive definite and

$$\pi_\lambda((e^A)^r) = (\pi_\lambda(e^A))^r, \qquad r \in \mathbb{R},$$

where $(\pi_\lambda(e^A))^r$ is the usual rth power of the positive definite operator $\pi_\lambda(e^A) \in \operatorname{Aut} V_\lambda$. In particular $|\pi_\lambda((e^A)^r)| = |\pi_\lambda(e^A)|^r$. So for $r \in \mathbb{R}$,

$$\begin{aligned} |\pi_\lambda(e^{A/2}e^{B}e^{A/2})^r| &= |\pi_\lambda(e^{A/2}e^{B}e^{A/2})|^r \qquad (e^{A/2}e^{B}e^{A/2} \in P) \\ &= |\pi_\lambda(e^{A}e^{B})|^r \\ &= |\pi_\lambda(e^{A})\pi_\lambda(e^{B})|^r, \end{aligned}$$

and

$$|\pi_\lambda(e^{rA/2}e^{rB}e^{rA/2})| = |\pi_\lambda(e^{rA}e^{rB})| = |(\pi_\lambda(e^A))^r(\pi_\lambda(e^B))^r|.$$

Since the operators $\pi_\lambda(e^A)$ and $\pi_\lambda(e^B)$ are positive definite, by Theorem 4.1 (2) and Remark 4.2,

$$\begin{aligned} |\pi_\lambda(e^{A/2}e^{B}e^{A/2})^r| &\le |\pi_\lambda(e^{rA/2}e^{rB}e^{rA/2})|, & r \ge 1, \\ |\pi_\lambda(e^{A/2}e^{B}e^{A/2})^r| &\ge |\pi_\lambda(e^{rA/2}e^{rB}e^{rA/2})|, & 0 \le r \le 1. \end{aligned}$$

By Theorem 2.1, the desired relations then follow.

Now $(e^{A/2}e^{B}e^{A/2})^r \in P$ since $e^{A/2}e^{B}e^{A/2} \in P$. Clearly $e^{rA/2}e^{rB}e^{rA/2} \in P$. Thus $(e^{A/2}e^{B}e^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$ in P and thus are hyperbolic [14, Proposition 6.2] and by [14, Theorem 6.1], the desired inequalities follow. □

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