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SOME INEQUALITIES FOR THE EXPONENTIALS

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Abstract. Let $|||\cdot|||$ be any give unitarily invariant norm. We generalize, in the context of semisimple Lie group, the inequalities (1) $|||e^A||| \le |||e^{\mathrm{Re}\, A}|||$ for all complex matrices A, where $\mathrm{Re}\, A$ denotes the Hermitian part of A, and (2) $|||e^{A+B}||| \le |||e^A e^B|||$ where A and B are $n\times n$ Hermitian matrices. The inequalities of Weyl, Ky Fan, Golden-Thompson, Lenard-Thompson, Cohen, and So-Thompson are recovered from the main results. Araki's relation on $(e^{A/2}e^Be^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$, where A, B are Hermitian and $r \in R$, is generalized.

1. Introduction

It is known [3, Theorem IX.3.1, Theorem IX.3.7] that for any unitarily invariant norm $|||\cdot||| : \mathbb{C}_{n\times n} \to \mathbb{R}$,

$$|||e^A||| \le |||e^{\mathrm{Re}\, A}|||, \quad A \in \mathbb{C}_{n\times n}, \tag{1.1}$$

$$|||e^{A+B}||| \le |||e^A e^B|||, \quad A, B \in \mathbb{C}_{n\times n} \text{ are Hermitian}, \tag{1.2}$$

where $\mathrm{Re}\, A$ denotes the Hermitian part of $A \in \mathbb{C}_{n\times n}$. Another result in Bhatia's Matrix Analysis [3, Theorem IX.3.5] implies that for any irreducible representation π of the general linear group $GL(n,\mathbb{C})$ (indeed it is sufficient to consider the semisimple $SL(n,\mathbb{C})$ with an appropriate scaling),

$$|\pi(e^{A+B})| \le |\pi(e^{\mathrm{Re}\, A}e^{\mathrm{Re}\, B})|, \quad A, B \in \mathbb{C}_{n\times n}, \tag{1.3}$$

where $|X|$ denotes the spectral radius (the maximum modulus of the eigenvalues) of the linear map X. In particular, when A, B are Hermitian, by considering the representation $\mathrm{tr} : GL(n,\mathbb{C}) \to \mathbb{C}$, we have the famous Golden-Thompson inequality [6, 16]

$$\mathrm{tr}\, e^{A+B} \le \mathrm{tr}\, (e^A e^B), \qquad A, B \text{ Hermitian}, \tag{1.4}$$

since the eigenvalues e^Ae^B are those of $e^{A/2}e^Be^{A/2}$ which is positive definite and thus are positive. See [14, 17, 1, 2] for some generalizations of Golden-Thompson's inequality. Indeed, Bhatia [3, p.259] defines a class of functions, called the class $\mathcal{T}$, and the notion comes out from a result of Thompson [17, Lemma 6].

Definition 1.1. A continuous function $f : \mathbb{C}_{n\times n} \to \mathbb{C}$ is said to be in the class $\mathcal{T}$ if it satisfies

(1) $f(XY) = f(YX)$ for all $X, Y \in \mathbb{C}_{n\times n}$,
(2) $|f(X^{2m})| \le f((XX^*)^m)$ for all $X \in \mathbb{C}_{n\times n}$, $m = 1, 2, \ldots$.

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Theorem 1.2. ([3, Theorem IX.3.5]) *If $f \in \mathcal{T}$, then for all $A, B \in \mathbb{C}_{n\times n}$,*

$$|f(e^{A+B})| \le f(e^{\mathrm{Re}\, A} e^{\mathrm{Re}\, B}).$$

Since XY and YX have the same eigenvalues, counting multiplicities, and the spectral radius of X is less than or equal to the operator norm of X, the spectral radius is an element of $\mathcal{T}$. Thus (1.3) follows from Theorem 1.2. However unitarily invariant norms generally fail to be in class $\mathcal{T}$. A quick example: consider the operator norm $\|\cdot\|$ which is clearly unitarily invariant and

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Certainly the first criterion of $\mathcal{T}$ is not satisfied for $\|\cdot\|$.

Though the appearance of (1.3) differs from that of (1.1) and (1.2), they can be derived from a pre-order order of Kostant seminal paper [13].

After the preliminary materials are introduced in Section 2, we extend in Section 3 the inequalities (1.1), (1.2) and (1.3) in the context of semisimple Lie group. In a sequence of remarks, we show how to derive from Theorem 3.1 the inequalities of

(1) Weyl [3] (the moduli of the eigenvalues of A are log majorized by the singular values of $A \in \mathbb{C}_{n\times n}$),
(2) Ky Fan [3] (the real parts of the eigenvalues of A are majorized by the real singular values of $A \in \mathbb{C}_{n\times n}$),
(3) Lenard-Thompson [14, 17] ($|||e^{A+B}||| \le |||e^{A/2}e^{B}e^{A/2}|||$, $A, B \in \mathbb{C}_{n\times n}$ Hermitian),
(4) Cohen [4] (the eigenvalues of the positive definite part of e^X (with respect to polar decomposition) are log majorized by the eigenvalues of $e^{\mathrm{Re}\, A}$, where $A \in \mathbb{C}_{n\times n}$),
(5) So-Thompson [15] (the singular values of e^A are weakly log majorized by the exponentials of the singular values of $A \in \mathbb{C}_{n\times n}$).

In Section 4 we extend, in the context of Lie group, Araki's result [1] on the relation of the two matrices $(e^{A/2}e^{B}e^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$ where $A, B \in \mathbb{C}_{n\times n}$ are Hermitian, $r \ge 0$. In the last section, the notion of class $\mathcal{T}$ functions is extended on the group level (will be called Thompson functions) and related inequalities are obtained.

2. Preliminaries

We recall some basic notions and results in [13]. Let $\mathfrak{g}$ be a real semisimple Lie algebra. Let G be any Lie group having $\mathfrak{g}$ as its Lie algebra. An element $X \in \mathfrak{g}$ is called real semisimple (nilpotent) if ad X is diagonalizable over $\mathbb{R}$ (ad X is nilpotent, respectively). An element $g \in G$ is called hyperbolic (unipotent) if $g = \exp(X)$ where $X \in \mathfrak{g}$ is real semisimple (nilpotent respectively). An element $g \in G$ is elliptic if Adg is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition [13, Proposition 2.1] for G asserts that each $g \in \mathfrak{g}$ can be uniquely written as

$$g = ehu,$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the analytic group of $\mathfrak{k}$ so that $\mathrm{Ad}(K)$ is a maximal compact subgroup of $\mathrm{Ad}(G)$. Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal Abelian subalgebra of $\mathfrak{g}$ in $\mathfrak{p}$. Then $A := \exp \mathfrak{a}$ is the analytic subgroup of $\mathfrak{a}$. Let W be the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. The Weyl group operates naturally in $\mathfrak{a}$ and A and the isomorphism $\exp : \mathfrak{a} \to A$ is a W-isomorphism.

For each real semisimple $X \in \mathfrak{g}$ (hyperbolic $h \in G$) let

$$c(X) = \mathrm{Ad}(G)X \cap \mathfrak{a}, \quad C(h) = \{ghg^{-1} : g \in G\} \cap A$$

denote the set of all elements in $\mathfrak{a}$ (A) which are conjugate to X (h, respectively). It turns out that $X \in \mathfrak{g}$ ($h \in G$, $e \in G$) is real semisimple (hyperbolic, elliptic) if and only if it is conjugate to an element in $\mathfrak{a}$ (A, K, respectively) [13, Proposition 2.3 and 2.4]. Thus $c(X)$ and $C(h)$ are single W-orbits in $\mathfrak{a}$ and A respectively. Moreover

$$C(\exp(X)) = \exp(c(X)).$$

Denote by $\mathrm{conv}\, W(X)$ the convex hull of the Weyl group orbit $c(X) \subset \mathfrak{a}$.

When $g \in G$ is arbitrary, define

$$C(g) := C(h(g)),$$

where $h(g)$ is the hyperbolic component of g and

$$\mathcal{A}(g) := \exp(\mathrm{conv}\, W(\log h(g))).$$

(For a hyperbolic $g \in G$, the real semisimple X such that $e^X = g$ is unique, and we write $\log g = X$, since $\mathrm{Ad}(e^X) = e^{\mathrm{ad}\, X}$ and the restriction of the usual matrix exponential map $e^A = \sum_{n=1}^{\infty} \frac{A^n}{n!}$ on the set of diagonalizable matrices over $\mathbb{R}$ is one-to-one). So $\mathcal{A}(g) \subset A$ and is invariant under the Weyl group. It is the "convex hull" of $C(g)$ in the multiplicative sense. Given $f, g \in G$, we say that $f \prec g$ if

$$\mathcal{A}(f) \subset \mathcal{A}(g),$$

or equivalently

$$C(f) \subset \mathcal{A}(g).$$

Notice that $\prec$ is a pre-order on G and $\mathcal{A}(\ell g \ell^{-1}) = \mathcal{A}(g)$ since $h(\ell g \ell^{-1}) = \ell h(g) \ell^{-1}$ for all $\ell \in G$, and is a partial order on the equivalence classes of hyperbolic elements under the conjugation of G. The order $\prec$ is different from Thompson's pre-order [17] on $SL(n, \mathbb{C})$ which simplifies the one made by Lenard [14] (The orders of Lenard and Thompson agree on the space of positive definite matrices).

Example 2.1. Let $G = SL(n, \mathbb{R})$ with $K = SO(n)$ and $A \subset SL(n, \mathbb{R})$ consists of positive diagonal matrices. Viewing $g \in SL(n, \mathbb{R})$ as an element in $\mathfrak{gl}(n, \mathbb{R})$, the additive Jordan decomposition [11, p.153] for $\mathfrak{gl}(n, \mathbb{R})$ yields

$$g = s + n_1$$

($s \in SL(n, \mathbb{R})$ semisimple, that is, diagonalizable over $\mathbb{C}$, $n_1 \in \mathfrak{sl}(n, \mathbb{R})$ nilpotent and $sn_1 = n_1 s$). Moreover these conditions determine s and n_1 completely [10, Proposition 4.2]. Put $u = 1 + s^{-1} n_1 \in SL(n, \mathbb{R})$ and we have the multiplicative Jordan decomposition

$$g = su,$$

where s is semisimple, u is unipotent, and $su = us$. By the uniqueness of the additive Jordan decomposition, s and u are also completely determined. Since s is diagonalizable,

$$s = eh,$$

where e is elliptic, h is hyperbolic, $eh = he$, and these conditions completely determine e and h. The decomposition can be obtained by observing that there is $k \in SL(n,\mathbb{C})$ such that

$$k^{-1}sk = s_1 I_{r_1} \oplus \cdots \oplus s_m I_{r_m},$$

where $s_1 = e^{i\xi_1}|s_1|, \ldots, s_m = e^{i\xi_m}|s_m|$ are the distinct eigenvalues of s with multiplicities $r_1, \ldots, r_m$ respectively. Set

$$e := k(e^{i\xi_1} I_{r_1} \oplus \cdots \oplus e^{i\xi_m} I_{r_m})k^{-1}, \quad h := k(|s_1| I_{r_1} \oplus \cdots \oplus |s_m| I_{r_m})k^{-1}.$$

If $s = e'h'$ with $e'h' = h'e'$, e' elliptic and h' hyperbolic, then s, e' and h' are simultaneously diagonalizable over $\mathbb{C}$ and hence for some $k' \in SL(n,\mathbb{C})$, $k'^{-1}sk' = s_1 I_{r_1} \oplus \cdots \oplus s_m I_{r_m}$,

$$e' = k'(e^{i\xi_1} I_{r_1} \oplus \cdots \oplus e^{i\xi_m} I_{r_m})k'^{-1}, \quad h' = k'(|s_1| I_{r_1} \oplus \cdots \oplus |s_m| I_{r_m})k'^{-1}.$$

Thus the first r_1 columns of k' form a basis for the eigenspace of s associated with the eigenvalue $s_1, \ldots\ldots$, and the last r_m columns of k' form a basis for the eigenspace of s associated with the eigenvalue s_m. So $k' = kB$ where $B \in \mathbb{C}_{r_1 \times r_1} \oplus \cdots \oplus \mathbb{C}_{r_m \times r_m}$ and thus $e' = e$ and $h = h$. Since

$$ehu = g = ugu^{-1} = ueu^{-1}uhu^{-1}u,$$

the uniqueness of s, u, e and h implies e, u and h commute. Since g is fixed under complex conjugation, the uniqueness of e, h and u imply $e, h, u \in SL(n,\mathbb{R})$ [7, p.431]. Thus $g = ehu$ is the complete multiplicative Jordan decomposition for $SL(n,\mathbb{R})$. The eigenvalues of h are simply the moduli of the eigenvalues of s and thus of g. We have similar decomposition for $SL(n,\mathbb{C})$.

Let $\mathfrak{sl}(n,\mathbb{R}) = \mathfrak{so}(n) + \mathfrak{p}$ be the fixed Cartan decomposition of $\mathfrak{sl}(n,\mathbb{R})$, that is, $\mathfrak{k} = \mathfrak{so}(n)$ and $\mathfrak{p}$ is the space of traceless real symmetric matrices. So $K = SO(n)$. Let $\mathfrak{a} \subset \mathfrak{p}$ be the maximal Abelian subalgebra of $\mathfrak{sl}(n,\mathbb{R})$ in $\mathfrak{p}$ containing the diagonal matrices. So the analytic group A of $\mathfrak{a}$ is the group of positive diagonal matrices of determinant 1. The Weyl group W of $(\mathfrak{a}, \mathfrak{g})$ is the full symmetric group S_n [12] which acts on A and $\mathfrak{a}$ by permuting the diagonal entries of the matrices in A and $\mathfrak{a}$. Now

$$C(f) = \{\mathrm{diag}\,(|\alpha_{\sigma(1)}|, \cdots, |\alpha_{\sigma(n)}|) : \sigma \in S_n\},$$

where $\alpha_1, \ldots, \alpha_n$ denote the eigenvalues of $f \in SL(n,\mathbb{C})$. So

$$c(\log h(f)) = \{\mathrm{diag}\,(\log|\alpha_{\sigma(1)}|, \cdots, \log|\alpha_{\sigma(n)}|) : \sigma \in S_n\}.$$

We will arrange them in such a way that $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$. So $f \prec g$, $f, g \in SL(n,\mathbb{R})$ means that the $\log h(f))$ is an element of the convex hull of the single W-orbit $c(\log h(g))$. Thus $\log|\alpha|$ is majorized by $\log|\beta|$ [3, p.33], denoted by $|\alpha| \prec_{\log} |\beta|$ which is called log majorization in [2], where β's are the eigenvalues of g. In other words, $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$, are multiplicatively majorized by

$|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_n|$, that is,

$$\begin{aligned}\prod_{i=1}^{k} |\alpha_i| &\leq \prod_{i=1}^{k} |\beta_i|, \quad k = 1, \dots, n-1,\\ \prod_{i=1}^{n} |\alpha_i| &= \prod_{i=1}^{n} |\beta_i|.\end{aligned}$$

On the other hand, one may deduce the above inequalities as necessary conditions for $f \prec g$ via Theorem 2.3 by considering the natural representation of $SL(n,\mathbb{R})$ on $V_\lambda = \mathbb{R}^n$ and the kth exterior powers $\wedge^k f$, $k = 1, \dots, n$. These would yield $\prod_{i=1}^k |\alpha_i| \leq \prod_{i=1}^k |\beta_i|$, $k = 1, \dots, n$. Then consider the representation $A \mapsto (\det A)^{-1}$ to have the equality. Same results hold for $SL(n,\mathbb{C})$.

Remark 2.2. In the above example, the pre-order $\prec$ in $SL(n,\mathbb{R}) \subset SL(n,\mathbb{C})$ coincides with that in $SL(n,\mathbb{C})$ since the Weyl groups are identical. But it is pointed out in [13, Remark 3.1.1] that the pre-order $\prec$ is not necessarily the same as the pre-order on the semisimple G that would be induced by a possible embedding of G in $SL(n,\mathbb{C})$ for some n.

We denote by $\hat{G}$ the index set of the irreducible representations of G, $\pi_\lambda : G \to \mathrm{Aut}\,(V_\lambda)$ a fixed representation in the class corresponding to $\lambda \in \hat{G}$, $|\pi_\lambda(g)|$ the spectral radius of the automorphism $\pi_\lambda(g) : V_\lambda \to V_\lambda$ where $g \in G$, that is, the maximum modulus of the eigenvalues of $\pi_\lambda(g)$, and χ_λ the character of π_λ. The following nice result of Kostant describes the pre-order $\prec$ via the irreducible representations of G and plays an important role in the coming sections.

Theorem 2.3. (Kostant [13, Theorem 3.1]) *Let $f, g \in G$. Then $f \prec g$ if and only if $|\pi_\lambda(f)| \leq |\pi_\lambda(g)|$ for all $\lambda \in \hat{G}$, where $|\cdot|$ denotes the spectral radius.*

3. The Main results

Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. For each $X \in \mathfrak{g}$, write $X = X_\mathfrak{k} + X_\mathfrak{p}$ where $X_\mathfrak{k} \in \mathfrak{k}$ and $X_\mathfrak{p} \in \mathfrak{p}$.

Theorem 3.1. *Let $\mathfrak{g}$ be a real semisimple Lie algebra. Let $X, Y \in \mathfrak{g}$ and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Then for any $n \geq 1$ and $g \in G$,*

$$g^{2n} \prec (g^*)^n g^n \prec (g^* g)^n,$$

and

$$e^{X+Y} \prec e^{-\theta(X+Y)/2} e^{(X+Y)/2} \prec e^{X_\mathfrak{p}} e^{Y_\mathfrak{p}},$$

where θ is the Cartan involution of $\mathfrak{g}$ with respect to the given Cartan decomposition.

By setting $Y = X$, we have

Corollary 3.2. Let $X \in \mathfrak{g}$. Then $e^X \prec e^{-\theta X/2} e^{X/2} \prec e^{X_\mathfrak{p}}$.

Proof. of Theorem 3.1 Let $\theta \in \mathrm{Aut}\,(\mathfrak{g})$ be the Cartan involution of $\mathfrak{g}$, that is, θ is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. Set $P = e^\mathfrak{p}$. We have the (global) Cartan decomposition

$$G = KP.$$

Then θ induces an automorphism Θ of G such that the differential of Θ at the identity is θ [12, p.387]. Explicitly

$$\Theta(kp) = kp^{-1}, \quad k \in K,\ p \in P.$$

For any $g \in G$ let

$$g^* := \Theta(g^{-1}).$$

If $g = kp$, the polar decomposition of $g \in G$, then

$$g^* = \Theta(p^{-1}k^{-1}) = \Theta(p^{-1})k^{-1} = pk^{-1},$$

and hence $g^*g = p^2 \in P$, since the centralizer $G^\Theta = \{g \in G : \Theta(g) = g\}$ coincides with K [12, p.305]. So

$$g^* := \Theta(g^{-1}) = (\Theta(g))^{-1}, \quad (g^*)^* = g, \quad (fg)^* = g^*f^*, \quad (g^*)^n = (g^n)^*,$$

for all $f, g \in G$, n positive integer. Since θ is the differential of Θ at the identity, we have [7, 110]

$$\Theta(e^A) = e^{\theta A},$$

for all $A \in \mathfrak{g}$. So

$$(e^A)^* = \Theta(e^{-A}) = e^{-\theta A}. \tag{3.1}$$

We now claim for any $g \in G$, and any natural number n,

$$g^{2n} \prec (g^*)^n g^n \prec (g^*g)^n. \tag{3.2}$$

The relation $g^{2n} \prec (g^*g)^n$ is known in [13, p.448] and we use similar idea (indeed the original idea can be found in [17] when $G = SL(n, \mathbb{C})$) to establish (3.2). We denote by $\Pi_\lambda : \mathfrak{g} \to \mathrm{End}\,(V_\lambda)$ the differential at the identity of the representation $\pi_\lambda : G \to \mathrm{Aut}\,(V_\lambda)$. So [7, p.110]

$$\exp \circ \Pi_\lambda = \pi_\lambda \circ \exp, \tag{3.3}$$

where the exponential function on the left is $\exp : \mathrm{End}\,(V_\lambda) \to \mathrm{Aut}\,(V_\lambda)$ and the one on the right side is $\exp : \mathfrak{g} \to G$. Now $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$ (direct sum) is a compact real form of $\mathfrak{g}_\mathbb{C}$ (the complexification of $\mathfrak{g}$). The representation $\Pi_\lambda : \mathfrak{g} \to \mathrm{End}\,(V_\lambda)$ naturally defines a representation $\mathfrak{u} \to \mathrm{End}\,(V_\lambda)$ of $\mathfrak{u}$, also denoted by Π_λ and vice versa. Let U be a simply connected Lie group of U [19, p.101] so that it is compact [5, Corollary 3.6.3]. There is a unique homomorphism $\hat{\pi}_\lambda : U \to \mathrm{Aut}\,(V_\lambda)$ such that the differential of $\hat{\pi}_\lambda$ at the identity is Π_λ [19, Theorem 3.27]. Thus there exists an inner product (we will assume that V_λ is endowed with this structure from now on) $\langle \cdot, \cdot \rangle$ on V_λ such that $\hat{\pi}_\lambda(u)$ is orthogonal for all $u \in U$. Differentiate the identity

$$(\hat{\pi}_\lambda(e^{tZ})X, \hat{\pi}_\lambda(e^{tZ})Y) = (X, Y),$$

for all $X, Y \in V_\lambda$ at $t = 0$ we have

$$(\Pi_\lambda(Z)X, Y) = -(X, \Pi_\lambda(Z)Y)$$

by (3.3). Thus $\Pi_\lambda(Z)$ is skew Hermitian for all $Z \in \mathfrak{u}$ [12, Proposition 4.6], [13, p.435]. Then $\Pi_\lambda(Z)$ is skew Hermitian if $Z \in \mathfrak{k}$ and is Hermitian if $Z \in \mathfrak{p}$. So $\pi_\lambda(z)$ is unitary if $z \in K$ and is positive definite if $z \in P$ by (3.3). Since each g can be written as $g = kp$, $k \in K$ and $p \in P$,

$$\begin{aligned}
\langle u, \pi_\lambda(g^*)v \rangle &= \langle u, \pi_\lambda(pk^{-1})v \rangle \\
&= \langle u, \pi_\lambda(p)\pi_\lambda(k^{-1})v \rangle \\
&= \langle \pi_\lambda(k)\pi_\lambda(p)u, v \rangle \\
&= \langle \pi_\lambda(g)u, v \rangle,
\end{aligned}$$

for all $u, v \in V_\lambda$. Thus

$$\pi_\lambda(g)^* = \pi_\lambda(g^*), \tag{3.4}$$

where $\pi_\lambda(g)^*$ denotes the Hermitian adjoint of $\pi_\lambda(g)$. Thus $\pi_\lambda(g^*g) = \pi_\lambda(g)^*\pi_\lambda(g) \in \mathrm{Aut}\,(V_\lambda)$ is a positive definite operator for all $g \in G$. Denote by $\|\pi_\lambda(g)\|$, $g \in G$, the operator norm of $\pi_\lambda(g)$. Thus

$$|\pi_\lambda(p)| = \|\pi_\lambda(p)\|, \quad \text{for all } p \in P.$$

Because of Theorem 2.3, to arrive at the claim (3.2) it suffices to show

$$|\pi_\lambda(g^{2n})| \le |\pi_\lambda((g^*)^n g^n)| \le |\pi_\lambda((g^*g)^n)|, \quad \text{for all } \lambda \in \hat{G}.$$

Now

$$\begin{aligned} |\pi_\lambda((g^*)^n g^n)| &= |\pi_\lambda((g^n)^* g^n)| \\ &= \|\pi_\lambda((g^n)^* g^n)\| \qquad (\pi_\lambda((g^n)^* g^n) \in \mathrm{Aut}\,(V_\lambda) \text{ is positive definite}) \\ &= \|\pi_\lambda(g^n)^* \pi_\lambda(g^n)\| \quad \text{by (3.4)} \\ &= \|\pi_\lambda(g^n)\|^2 \qquad (\|T\|^2 = \|T^*T\|). \end{aligned}$$

On the other hand,

$$\begin{aligned} |\pi_\lambda((g^*g)^n)| &= |\pi_\lambda(g^*g)|^n \\ &= \|\pi_\lambda(g^*g)\|^n \qquad (\pi_\lambda((g^*g) \in \mathrm{Aut}\,(V_\lambda) \text{ is positive definite}) \\ &= \|\pi_\lambda(g)^*\pi_\lambda(g)\|^n \\ &= \|\pi_\lambda(g)\|^{2n} \qquad (\|T\|^2 = \|T^*T\|) \\ &\ge \|\pi_\lambda(g^n)\|^2 \qquad (\|T^n\| \le \|T\|^n), \end{aligned}$$

where the inequality is due to the well known fact that the spectral radius is no greater than the operator norm. So we have $(g^*)^n g^n \prec (g^*g)^n$. Now

$$|\pi_\lambda((g^*)^n)g^n)| = |\pi_\lambda((g^n)^*)\pi_\lambda(g^n)| = \|\pi_\lambda(g^n)\|^2 \ge |\pi_\lambda(g^n)|^2 = |\pi_\lambda(g^{2n})|.$$

Hence $g^{2n} \prec (g^*)^n g^n$ and we just proved the claim.

By the first relation in (3.2), if $g = xy$ where $x, y \in G$, we have for any natural number m,

$$(xy)^{2^{m+1}} \prec (y^*x^*)^{2^m}(xy)^{2^m}.$$

Set $x = e^{X/2^m}$, $y = e^{Y/2^m}$, where $X, Y \in \mathfrak{g}$. We get

$$\begin{aligned} ((e^{X/2^m}e^{Y/2^m})^{2^m})^2 &\prec ((e^{Y/2^m})^*(e^{X/2^m})^*)^{2^m}(e^{X/2^m}e^{Y/2^m})^{2^m} \\ &= (e^{-\theta Y/2^m}e^{-\theta X/2^m})^{2^m}(e^{X/2^m}e^{Y/2^m})^{2^m} \end{aligned}$$

by (3.1). Since $\lim_{t\to\infty}(e^{X/t}e^{Y/t})^t = e^{X+Y}$ [7, p.115], and the relation $\prec$ remains valid as we take limits on both sides because the spectral radius is a continuous function on $\mathrm{Aut}\,(V_\lambda)$, we have $e^{2(X+Y)} \prec e^{-\theta(X+Y)}e^{(X+Y)}$. As a result

$$e^{X+Y} \prec e^{-\frac{1}{2}\theta(X+Y)}e^{\frac{1}{2}(X+Y)},$$

and we just established the first part of Theorem 3.1.

Let $g = e^{(X+Y)/n}$, $X, Y \in \mathfrak{g}$. By the second relation of (3.2),

$$(e^{-\theta(X+Y)/n})^n(e^{(X+Y)/n})^n \prec ((e^{-\theta(X+Y)/n}e^{(X+Y)/n}))^n.$$

So

$$e^{-\theta(X+Y)}e^{X+Y} \prec e^{2(X+Y)_\mathfrak{p}} = e^{2X_\mathfrak{p}+2Y_\mathfrak{p}} \prec e^{2X_\mathfrak{p}}e^{2Y_\mathfrak{p}},$$

where the last relation is established in [13, Theorem 6.3]. $\square$

Remark 3.3. Certainly, the statement $e^{X+Y} \prec e^{X_{\mathfrak{k}}}e^{Y_{\mathfrak{k}}}$ is not true by simply considering $G = SL(n,\mathbb{C})$ in which $K = SU(n)$ and $\mathfrak{k} = \mathfrak{su}(n)$. There $e^{X_{\mathfrak{k}}}e^{Y_{\mathfrak{k}}} \in SU(n)$ and we may pick $X, Y \in \mathfrak{sl}(n,\mathbb{C})$ such that $X+Y$ is nonzero Hermitian matrix with a positive eigenvalue. Viewing each $g \in SL(n,\mathbb{C})$ as a linear operator on $V_\lambda = \mathbb{C}^n$ (the natural representation of $SL(n,\mathbb{C})$), the spectral radius $|e^{X_{\mathfrak{k}}}e^{Y_{\mathfrak{k}}}| = 1$ but $|e^{X+Y}| > 1$.

Remark 3.4. (Cohen's result) When $G = GL(n,\mathbb{C})$, the relation $g^{*n}g^n \prec (g^*g)^n$ was established in [4] and $g^{2n} \prec (g^*g)^n$ was obtained in [17]. Kostant [13, Proof of Theorem 6.3] also proved $g^{2n} \prec (g^*g)^n$ and $e^{A+B} \prec e^Ae^B$, $A, B \in \mathfrak{p}$, for general G. The relation in Theorem 3.1

$$g^{*n}g^n \prec (g^*g)^n$$

is equivalent to

$$p(g^n) \prec (p(g))^n,$$

where $g = k(g)p(g)$ is the polar decomposition of $g \in G$. If we set $g = e^{X/n}$, then we have

$$p(e^X) \prec [p(e^{X/n})]^n, \qquad n = 1, 2, ...$$

Now $p(e^{X/n}) = ((e^{X/n})^* e^{X/n})^{1/2} = (e^{-\theta X/n}e^{X/n})^{1/2}$. So

$$\lim_{n\to\infty}[p(e^{X/n})]^n = \lim_{n\to\infty}[(e^{-\theta X/n}e^{X/n})^{1/2}]^n = e^{X_{\mathfrak{p}}},$$

and thus

$$p(e^X) \prec e^{X_{\mathfrak{p}}}$$

which is Cohen's result [4] when $G = SL(n,\mathbb{C})$ with appropriate scaling.

Remark 3.5. (Ky Fan's inequality and inequality (1.1))

Continuing with Example 2.1, for $A \in \mathfrak{sl}(n,\mathbb{C})$, the moduli of the eigenvalues of e^A are the exponentials of the real parts of the eigenvalues of A, counting multiplicities. The matrix $e^{\mathrm{Re}\,A}$ is positive definite. So the eigenvalues of $e^{\mathrm{Re}\,A}$ are indeed the singular values, and are the exponentials of the eigenvalues of $\mathrm{Re}\,A$. The eigenvalues of $\mathrm{Re}\,A$ are known as the real singular values of A, denoted by $\beta_1 \ge \cdots \ge \beta_n$. Denote the real parts of the eigenvalues of A by $\alpha_1 \ge \cdots \ge \alpha_n$. By Corollary 3.2 $e^A \prec e^{\mathrm{Re}\,A}$ which amounts to

$$\begin{aligned} \prod_{i=1}^k e^{\alpha_i} &\le \prod_{i=1}^k e^{\beta_i}, \quad i = 1, \ldots, n-1, \\ \prod_{i=1}^n e^{\alpha_i} &= \prod_{i=1}^n e^{\beta_i}, \end{aligned}$$

that is $e^\alpha \prec_{\log} e^\beta$. Thus, by taking log on the above relation, the relation $e^A \prec e^{\mathrm{Re}\,A}$ amounts to the usual majorization relation $\alpha \in \mathrm{conv}\, S_n\beta$, a well known result of Ky Fan [3, Proposition III.5.3]. From the second relation of Corollary 3.2, $e^Ae^{A^*} \prec e^{A+A^*}$ which amounts to the fact that the singular values of e^A (that is, the square roots of the eigenvalues of $e^Ae^{A^*}$) are multipicatively majorized, and hence weakly majorized [3, p.42], [2], by the singular values (also the eigenvalues) of the positive definite $e^{\mathrm{Re}\,A}$. Thus

$$|||e^A||| \le |||e^{\mathrm{Re}\,A}|||,$$

for all unitarily invariant norms $|||\cdot|||$ [3, Theorem IX.3.1] by Ky Fan Dominance Theorem [3, Theorem IV.2.2]. Thus we have (1.1).

Remark 3.6. (So-Thompson's inequality)

From $e^A e^{A^*} \prec e^{A+A^*}$, $A \in \mathbb{C}_{n\times n}$, So-Thompson inequalities [15, Theorem 2.1] asserts that

$$\prod_{i=1}^k s_i(e^A) \le \prod_{i=1}^k e^{s_i(A)}, \quad k = 1, \dots, n$$

can be derived via Fan-Hoffman inequalities [3, proposition III.5.1]

$$\lambda_i(\operatorname{Re} A) \le s_i(A), \quad i = 1, \dots, n,$$

where $s_1(A) \ge \cdots \ge s_n(A)$ denote the singular values of $A \in \mathbb{C}_{n\times n}$.

Remark 3.7. (Weyl's inequality and inequalities (1.2) and (1.3))

Let $A \in SL(n, \mathbb{C})$. By (3.4) $A^2 \prec A^*A$. By Example 2.1, $|\lambda^2(A)| \prec_{\log} |\lambda(A^*A)| = |s(A^*A)|$, that is,

$$|\lambda(A)| \prec_{\log} s(A).$$

By scaling and continuity argument, the log majorization remains valid for $A \in \mathbb{C}_{n\times n}$, that is, Weyl's inequality [3, p.43]. In the literature, Weyl's inequality is often proved via the kth exterior power once $|\lambda_1(A)| \le s_1(A)$ is established, for example [3, p.42-43]. Such an approach shares some favor of Theorem 2.3.

If $A, B \in C_{n\times n}$ are Hermitian, then e^A, e^B and e^{A+B} are positive definite. Though e^Ae^B is not positive definite in general, its eigenvalues, denoted by $\delta_1 \ge \cdots \ge \delta_n$, are positive since e^Ae^B and the positive definite $e^{A/2}e^Be^{A/2}$ share the same eigenvalues. Denote the eigenvalues of e^{A+B} by $\gamma_1 \ge \cdots \ge \gamma_n$. Thus γ is multiplicatively majorized by δ because of $e^{A+B} \prec e^Ae^B$ (Theorem 3.1). Notice that δ is also multiplicatively majorized by the singular values $s_1 \ge \cdots \ge s_n$ of e^Ae^B, by Weyl's inequality. Hence we have the weak majorization relation $\gamma \prec_w s$ [3, p.42] so that (1.2) follows. Finally (1.3) follows from Theorem 3.1 and Theorem 2.3.

Remark 3.8. (Lenard-Thompson's inequality) Lenard's result [14] together with [17, Theorem 2] imply that

$$|||e^{A+B}|||| \le |||e^{A/2}e^Be^{A/2}|||, \quad A, B \in \mathbb{C}_{n\times n} \text{ Hermitian}, \tag{3.5}$$

from which Golden-Thompson's result follows. It is because e^{A+B} and $e^{A/2}e^Be^{A/2}$ are positive definite and their traces are indeed the Ky Fan n-norm, that is, sum of singular values which is unitarily invariant. Indeed Lenard's result just asserts that any arbitrary neigborhood of e^{A+B} contains X such that $X \prec e^{A/2}e^Be^{A/2}$ [14, p.458] (It is weaker than (3.6)). By a limit argument and Thompson's argument, (3.5) follows. But the more basic question is whether (3.6) is true. Indeed

$$e^{A+B} \prec e^Ae^B, \quad A, B \in \mathfrak{p}$$

(Theorem 3.1) is a unified generalization of Golden-Thompson's inequality and (1.2) and (3.5) in the context of Lie group since

$$e^{A+B} \prec e^{A/2}e^Be^{A/2}, \quad A, B \in \mathfrak{p}. \tag{3.6}$$

Now (3.6) is true simply because $\pi_\lambda(e^Ae^B)$ and $\pi_\lambda(e^{A/2}e^Be^{A/2})$ have the same spectrum (by the fact that XY and YX have the same spectrum and π_λ is a representation) and thus have the same spectral radius. Then apply Theorem 2.3.

4. Extension of Araki's result

Araki's result [1] (actually it appears in the proof of the main Theorem [1, p.168-169]. Also see [9] for a short proof) asserts that if $A, B \in \mathbb{C}_{n\times n}$ Hermitian, then

$$(e^{A/2}e^Be^{A/2})^r \prec e^{rA/2}e^{rB}e^{rA/2}, \qquad r>1, \tag{4.1}$$

that amounts to

$$s((e^{A/2}e^Be^{A/2})^r) \prec_{\log} s(e^{rA/2}e^{rB}e^{rA/2}), \qquad r>1,$$

or equivalently

$$s((e^{qA/2}e^{qB}e^{qA/2})^{1/q}) \prec_{\log} s((e^{pA/2}e^{pB}e^{pA/2})^{1/p}), \qquad 0<q\le p.$$

Together with Lie-Trotter formula

$$e^{A+B} = \lim_{r\to 0}(e^{rA/2}e^{rB}e^{rA/2})^{1/r},$$

Golden-Thompson's result is strenghtened [2]:

$$|||e^{pA/2}e^{pB}e^{pB/2}|||$$

decreases down to $|||e^{A+B}|||$ as $p\downarrow 0$ for any unitarily invariant norm $|||\cdot|||$ on $\mathbb{C}_{n\times n}$ and in particular

$$\mathrm{tr}\ e^{A+B} \le \mathrm{tr}\ [e^{pA/2}e^{pB}e^{pB/2}]^{1/p}, \qquad p>0.$$

Araki's result also implies a result of Wang and Gong [18] (also see [3, Theorem IX.2.9]).

In order to extend (4.1) for general G, we need a result of Heinz [8] conerning two positive semidefinte operators. Indeed the orginal proof of Araki's result [1] also makes use of Heinz's result. Give two positive semidefinite operators A, B, the spectrum (counting multiplicities) $\lambda(AB) = \lambda(A^{1/2}BA^{1/2})$ and thus all eigenvalues of AB are positive. So the largest eigenvalue of AB, $\lambda_1(AB)$, is the spectral radius of AB. The first part of the following theorem is due to Heinz [8] (see [p.255-256] for two nice proofs of Heinz's result). The second part is proved via the Heinz's result in [3, Theorem IX.2.6] in a somewhat lengthly way.

Theorem 4.1. *The following two statements are equivalent and valid.*

(1) *(Heinz) For any two positive semidefinite operators A, B, $\|A^sB^s\| \le \|AB\|^s$, $0\le s\le 1$.*
(2) *For any two positive semidefinite operators A, B, $\lambda_1(A^sB^s) \le \lambda_1^s(AB)$, $0\le s\le 1$.*

Proof. We just establish the equivalence of the two statements. Since $\|T\| = \|T^*T\|^2$,

$$\|A^sB^s\| = \|(A^sB^s)A^sB^s\|^{1/2} = \|B^sA^{2s}B^s\|^{1/2} = \lambda_1^{1/2}(B^sA^{2s}B^s) = \lambda_1^{1/2}(A^{2s}B^{2s}),$$

and

$$\|AB\|^s = \|ABBA\|^{s/2} = \lambda_1^{s/2}(AB^2A) = \lambda_1^{s/2}(A^2B^2).$$

□

Remark 4.2. An equivalent statement to Heniz's result is: for any positive operators A, B, $\|A^tB^t\| \ge \|AB\|^t$ if $t\ge 1$, or equivalently $\lambda_1(A^tB^t) \ge \lambda_1^t(AB)$ [3, p.256-257].

For general G, the map $\exp : \mathfrak{p} \to P$ where $P := e^{\mathfrak{p}}$ is one-to-one since the map

$$(K, \mathfrak{p}) \to G, \quad (k, X) \mapsto ke^X$$

is a diffeomorphism [12, p.305], and thus $(e^A)^r := e^{rA} \in P$ where $r \in \mathbb{R}$. So $f^r, g^r \in P$, $f^r g^r$ (hyperbolic, since $f^r g^r$ is conjugate to $f^{r/2} g^r f^{r/2}$), $r \in \mathbb{R}$, are well defined for $f, g \in P$. The following is an extension of Heinz's result on the group level.

Theorem 4.3. *Let $f, g \in P$. Then*

$$\begin{array}{rcll} (fg)^t & \prec & f^t g^t, & t \geq 1, \\ (fg)^s & \prec & f^s g^s, & 0 \leq s \leq 1. \end{array}$$

Proof. Since each element e^A in P ($A \in \mathfrak{p}$) is of the form $e^{-\theta A/2} e^{A/2} = (e^A)^* e^A$ ($A = -\theta A$), $\pi_\lambda(e^A)$ is positive definite. Thus $\pi_\lambda(f), \pi_\lambda(g) \in \mathrm{Aut}\,(V_\lambda)$ are positive definite if $f, g \in P$. Suppose $0 \leq s \leq 1$. Then

$$\begin{array}{rcl} |\pi_\lambda((fg)^s)| = |\pi_\lambda(fg)|^s = |\pi_\lambda(f)\pi_\lambda(g)|^s & \geq & |\pi^s_\lambda(f)\pi^s_\lambda(g)| \\ & = & |(\pi_\lambda(f^s)\pi_\lambda(g^s)| = |(\pi_\lambda(f^s g^s)|, \end{array}$$

by Theorem 4.1 (2). Applying Theorem 2.3 to have the desired result $(fg)^s \prec f^s g^s$, $0 \leq s \leq 1$. The other relation is by Remark 4.2. ☐

When $A, B \in \mathfrak{p}$, the element $e^{A/2} e^B e^{A/2}$ is in P since it is of the form $g^* g$ where $g = e^{B/2} e^{A/2}$. Thus $(e^{A/2} e^B e^{A/2})^r \in P$, $r \in \mathbb{R}$ is well defined.

Theorem 4.4. *Let $A, B \in \mathfrak{p}$. Then*

$$\begin{array}{rcll} (e^{A/2} e^B e^{A/2})^r & \prec & e^{rA/2} e^{rB} e^{rA/2}, & r > 1, \\ e^{rA/2} e^{rB} e^{rA/2} & \prec & (e^{A/2} e^B e^{A/2})^r, & 0 \leq r \leq 1. \end{array}$$

Moreover, for all $\lambda \in \hat{G}$

$$\begin{array}{rcll} \chi_\lambda((e^{A/2} e^B e^{A/2})^r) & \leq & \chi_\lambda(e^{rA/2} e^{rB} e^{rA/2}), & r > 1, \\ \chi_\lambda(e^{rA/2} e^{rB} e^{rA/2}) & \leq & \chi_\lambda((e^{A/2} e^B e^{A/2})^r), & 0 \leq r \leq 1. \end{array}$$

Proof. Notice that $\pi_\lambda(e^A)$ is positive definite and

$$\pi_\lambda((e^A)^r) = (\pi_\lambda(e^A))^r, \qquad r \in \mathbb{R},$$

where $(\pi_\lambda(e^A))^r$ is the usual rth power of the positive definite operator $\pi_\lambda(e^A) \in \mathrm{Aut}\,(V_\lambda)$. In particular $|\pi_\lambda((e^A)^r)| = |\pi_\lambda(e^A)|^r$. So for $r \in \mathbb{R}$,

$$\begin{array}{rcll} |\pi_\lambda(e^{A/2} e^B e^{A/2})^r| & = & |\pi_\lambda(e^{A/2} e^B e^{A/2})|^r & (e^{A/2} e^B e^{A/2} \in P) \\ & = & |\pi_\lambda(e^A e^B)|^r & \\ & = & |\pi_\lambda(e^A)\pi_\lambda(e^B)|^r, & \end{array}$$

and

$$|\pi_\lambda(e^{rA/2} e^{rB} e^{rA/2})| = |\pi_\lambda(e^{rA} e^{rB})| = |(\pi_\lambda(e^A))^r (\pi_\lambda(e^B))^r|.$$

Since the operators $\pi_\lambda(e^A)$ and $\pi_\lambda(e^B)$ are positive definite, by Theorem 4.1 (2) and Remark 4.2,

$$\begin{array}{rcll} |\pi_\lambda(e^{A/2} e^B e^{A/2})^r| & \leq & |\pi_\lambda(e^{rA/2} e^{rB} e^{rA/2})|, & r \geq 1, \\ |\pi_\lambda(e^{A/2} e^B e^{A/2})^r| & \geq & |\pi_\lambda(e^{rA/2} e^{rB} e^{rA/2})|, & 0 \leq r \leq 1. \end{array}$$

By Theorem 2.3, the desired relations then follow.

Now $(e^{A/2}e^Be^{A/2})^r \in P$ since $e^{A/2}e^Be^{A/2} \in P$. Clearly $e^{rA/2}e^{rB}e^{rA/2} \in P$. Thus $(e^{A/2}e^Be^{A/2})^r$ and $e^{rA/2}e^{rB}e^{rA/2}$ in P and thus are hyperbolic [13, Proposition 6.2] and by [13, Theroem 6.1], the desired inequalities follow. □

5. Thompson functions

Definition 5.1. [3, 14] A continuous function $\phi : G \to \mathbb{C}$ is a Thompson function if it satisfies

(1) $\phi(fgf^{-1}) = \phi(g)$ for all $f, g \in G$, that is, ϕ is a class function with respect to conjugation.
(2) $|\phi(g^{2m})| \le \phi((g^*g)^m)$ for all $g \in G$, $m = 1, 2, \ldots$.

Notice that we define Thompson functions on the group G instead of the Lie algebra $\mathfrak{g}$. In the case $G = GL(n,\mathbb{C})$ (reductive), class $\mathcal{T}$ functions are defined on $\mathfrak{gl}(n,\mathbb{C})$ [3, 17] and $GL(n,\mathbb{C})$ just happens to be a subset of its Lie algebra $\mathfrak{gl}(n,\mathbb{C})$ but it is not necessarily true for general semi-simple or reductive Lie groups.

Theorem 5.2. *Let $\phi : G \to \mathbb{C}$ be a Thompson function. Then*

(1) *$\phi(e^A) \ge 0$ if $A \in \mathfrak{p}$, and*
(2) *$|\phi(e^{A+B})| \le \phi(e^{A_\mathfrak{p}}e^{B_\mathfrak{p}})$ for all $A, B \in \mathfrak{g}$. Thus $|\phi(e^A)| \le \phi(e^{A_\mathfrak{p}})$ if $A \in \mathfrak{g}$, and $0 \le \phi(e^{A+B}) \le \phi(e^Ae^B)$ if $A, B \in \mathfrak{p}$.*

Proof. (1) $\phi(e^A) = \phi(e^{A/2}e^{A/2}) = \phi(e^{A/2}e^{A^*/2})$ since $A \in \mathfrak{p}$. Then apply the second property of ϕ.

(2) The first condition of ϕ is equivalent to $\phi(fg) = \phi(gf)$, for all $f, g \in G$. We repeat the argument in Bhatia [3, p.260] word for word. For any positive integer m, by the properties of ϕ, we have for all $f, g \in G$,

$$|\phi((fg)^{2^m})| \le \phi(((fg)^*(fg))^{2^{m-1}}) = \phi((g^*f^*fg)^{2^{m-1}}) = \phi((f^*fgg^*)^{2^{m-1}}).$$

Repeat the argument to obtain

$$|\phi((fg)^{2^m})| \le \phi(((f^*f)^2(gg^*)^2)^{2^{m-2}}) \le \cdots \le \phi((f^*f)^{2^{m-1}}(gg^*)^{2^{m-1}}).$$

Set $f = e^{A/2^m}$ and $g = e^{B/2^m}$. Thus

$$|\phi((e^{A/2^m}e^{B/2^m})^{2^m})| \le \phi((e^{A^*/2^m}e^{A/2^m})^{2^{m-1}}(e^{B^*/2^m}e^{B/2^m})^{2^{m-1}}).$$

Applying the Lie product formula we conclude

$$|\phi(e^{A+B})| \le \phi(e^{A_\mathfrak{p}}e^{B_\mathfrak{p}}),$$

and the rest follow immediately.

□

See [3, Exercise IX.3.3] for some examples of Thompson functions on $SL(n,\mathbb{C})$ by switching $\mathbb{C}_{n\times n}$ to $SL(n,\mathbb{C})$. With some scaling, the particular case $\phi(g) := \operatorname{tr} g$, $g \in SL(n,\mathbb{C})$ yields Golden-Thompson inequality. For general G, the character $\chi_\lambda := \operatorname{tr} \pi_\lambda : G \to \mathbb{C}$ is a Thompson function since

$$|\operatorname{tr} \pi_\lambda(g^{2m})| = |\operatorname{tr} \pi_\lambda(g^2)|^m \le \operatorname{tr} \pi_\lambda(g^*g)^m = \operatorname{tr} \pi_\lambda((g^*g)^m),$$

by Cauchy-Schwarz's inequality. Thus we have

Corollary 5.3. Given $\lambda \in \hat{G}$, the character $\chi_\lambda : G \to \mathbb{C}$ is a Thompson function. Hence

(1) $0 \le \chi_\lambda(e^A)$, $A \in \mathfrak{p}$.

(2) If $X, Y \in \mathfrak{g}$, then

$$|\chi_\lambda(e^{X+Y})| \le \chi_\lambda(e^{X_\mathfrak{p}} e^{Y_\mathfrak{p}}),$$

for all $\lambda \in \hat{G}$, where χ_λ denotes the character of π_λ. In addition if e^{X+Y} is hyperbolic, $0 \le \chi_\lambda(e^{X+Y}) \le \chi_\lambda(e^{X_\mathfrak{p}} e^{Y_\mathfrak{p}})$. Moreover (i) $|\phi(e^X)| \le |\phi(e^{X_\mathfrak{p}})|$, $X \in \mathfrak{p}$, and (ii) $|\chi_\lambda(e^{A+B})| \le \chi_\lambda(e^A e^A)$ if $A, B \in \mathfrak{p}$.

Corollary 5.3 (1) is trivial since $\pi_\lambda(e^A)$ is positive definite if $A \in \mathfrak{p}$. Corollary 5.3 (2)(ii) is contained in [13, Theorem 6.3].

When $X + Y$ is real semisimple, that is, e^{X+Y} is hyperbolic and is conjugate to $e^Z \in e^\mathfrak{p}$, $Z \in \mathfrak{a} \subset \mathfrak{p}$. So $\pi_\lambda(e^{X+Y})$ is similar to the positive definite operator $\pi_\lambda(e^Z)$ and hence $|\chi_\lambda(e^{X+Y})| = \chi_\lambda(e^{X+Y})$. Then $0 < \chi_\lambda(e^{X+Y}) \le \chi_\lambda(e^{X_\mathfrak{p}} e^{Y_\mathfrak{p}})$.

Example 5.4. Let $G = SL(2, \mathbb{R})$. Let

$$A := \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$$

which is real semisimple, that is, diagonalizable over $\mathbb{R}$. We can decompose $A = X + Y$, $X, Y \in \mathfrak{sl}(2, \mathbb{R})$, in various ways. For examples,

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{Re} X = X, \quad \operatorname{Re} Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

or

$$X = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad \operatorname{Re} X = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \operatorname{Re} Y = Y.$$

The inequality $\chi_\lambda(e^{X+Y}) \le \chi_\lambda(e^{\operatorname{Re} X} e^{\operatorname{Re} Y})$, $\lambda \in \hat{G}$, holds for all such decompositions.

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