

# UNITARY SIMILARITY TO A COMPLEX SYMMETRIC MATRIX AND ITS EXTENSION TO ORTHOGONAL SYMMETRIC LIE ALGEBRAS

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ABSTRACT. We present some new characterizations of unitary similarity of a square complex matrix to a symmetric matrix. Our approach uses the singular value decomposition. A result of Vermeer is extended in the context of an orthogonal symmetric Lie algebra of the compact type.

## 1. INTRODUCTION

Every  $n \times n$  complex matrix is similar to a complex symmetric matrix [5, Theorem 4.4.9], but it is often difficult to tell whether or not a given matrix is unitarily similar to a complex symmetric matrix. See examples in [2]. When  $n = 2$ , each  $A \in \mathbb{C}_{2 \times 2}$  is unitarily similar to a symmetric matrix [7, p.477]. Vermeer obtained the following characterizations:

**Theorem 1.1.** [10, Theorem 3] Let  $A \in \mathbb{C}_{n \times n}$ . The following statements are equivalent.

- (1)  $A$  is unitarily similar to a complex symmetric matrix.
- (2) There is a symmetric unitary matrix  $U$  such that  $UAU^*$  is symmetric.
- (3) There is a symmetric unitary matrix  $U$  and a symmetric matrix  $S$  such that  $A = SU$ .
- (4) There is a symmetric unitary matrix  $V$  such that  $VAV^* = A^T$ .

When  $n \leq 7$ , the *symmetric* restriction in statement (4) can be removed [3]. Vermeer [10] studied this problem over  $\mathbb{R}$  and provided an example to show that the symmetric restriction in (4) is necessary if  $n \geq 8$ . Tener [9] gave some equivalent conditions in terms of the Hermitian decomposition  $A = \frac{1}{2}(A + A^*) + \frac{1}{2}(A - A^*)$  of  $A$ . Garcia et

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al. [2] obtained some criteria based on the singular value decomposition under the assumption that the singular values are distinct.

In Section 2, we give some new characterizations. In Section 3, we extend Theorem 1.1 in the context of an orthogonal symmetric Lie algebra of the compact type. Explicit statements for classical simply connected Riemannian globally symmetric spaces are worked out as examples. Theorem 1.1 essentially corresponds to the symmetric space  $SU(n)/SO(n)$ .

## 2. SOME CHARACTERIZATIONS

Let  $U(n)$  and  $O(n)$  denote the unitary group and orthogonal group respectively. Let  $A \in \mathbb{C}_{n \times n}$  and let  $A = V\Sigma U^*$  be a *singular value decomposition* (SVD) of  $A$ , where  $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$  and  $\sigma_1 \geq \dots \geq \sigma_n$  are the singular values of  $A$ . Notice that  $U, V \in U(n)$  are not uniquely determined. However, if the singular values are distinct, then  $U$  and  $V$  are respectively unique up to post-multiplication by the same diagonal unitary matrix. Clearly  $A^*AU = U\Sigma^2$  and  $AA^*V = V\Sigma^2$ .

The following is known as the Autonne decomposition [5, Corollary 4.4.4], [6, p.136].

**Lemma 2.1.** If  $A \in \mathbb{C}_{n \times n}$  is symmetric, then there is a  $U \in U(n)$  such that  $A = U\Sigma U^T$ . In particular, if  $A$  is symmetric unitary, then  $A = UU^T$  for some  $U \in U(n)$ .

The following characterization does not assume *distinct* singular values as required in [2].

**Theorem 2.2.** Let  $A \in \mathbb{C}_{n \times n}$  and let  $\Sigma = \sigma_1 I_{n_1} \oplus \dots \oplus \sigma_m I_{n_m}$ , where  $\sigma_1 > \dots > \sigma_m$  are the distinct singular values of  $A$  with multiplicities  $n_1, \dots, n_m$  respectively. The following statements are equivalent.

- (1)  $A$  is unitarily similar to a complex symmetric matrix.
- (2) There is a SVD  $A = Y\Sigma X^*$  such that  $X^*Y$  is symmetric.
- (3) For any SVD  $A = V\Sigma U^*$ , there are block diagonal unitary matrices  $Q := Q_1 \oplus \dots \oplus Q_m$  and  $Q' := Q_1 \oplus \dots \oplus Q_{m-1} \oplus Q'_m$ , conformal to  $\Sigma$ , such that  $(UQ)^*VQ'$  is symmetric. If  $\sigma_m > 0$ , then  $Q = Q'$ .

*Proof.* (1)  $\Rightarrow$  (2). Suppose that  $A$  is unitarily similar to a complex symmetric matrix  $S$ , i.e.,  $A = W^*SW$  with  $W \in U(n)$ . Then  $S = Z^T\Sigma Z$  for some  $Z \in U(n)$  by Lemma 2.1. Thus  $A = (W^*Z^T)\Sigma(ZW)$  is a SVD of  $A$  with  $(ZW)(W^*Z^T) = ZZ^T$  symmetric.

(2)  $\Rightarrow$  (3). Since  $U, X \in U(n)$  and the columns of  $U$  and  $X$  are eigenvectors of  $A^*A$  corresponding to the eigenvalues  $\sigma_i^2$ 's, we have

$X = UQ$  for some  $Q := Q_1 \oplus \cdots \oplus Q_m$ , where  $Q_j \in U(n_j)$ ,  $j = 1, \dots, m$ . Similarly  $Y = VQ'$  for some  $Q' := Q'_1 \oplus \cdots \oplus Q'_m$ , where  $Q'_j \in U(n_j)$ ,  $j = 1, \dots, m$ . Thus  $(UQ)^*(VQ') = X^*Y$  is symmetric. Notice that

$$A = Y\Sigma X^* = VQ'\Sigma(UQ)^* = V\Sigma Q'Q^*U^*$$

so that  $\Sigma Q'Q^* = V^*AU = \Sigma$ . Since  $\sigma_{m-1} > 0$ ,  $Q'_i Q_i^* = I_{n_i}$ , i.e.,  $Q'_i = Q_i$ , for  $i = 1, \dots, m-1$ . In addition, if  $\sigma_m > 0$ , then  $Q'_m = Q_m$  as well, i.e.,  $Q' = Q$ .

(3)  $\Rightarrow$  (1). Notice that  $Q'\Sigma Q^* = \Sigma$  so that

$$A = V\Sigma U^* = (VQ')\Sigma(UQ)^*$$

is a SVD of  $A$ . Since  $(UQ)^*VQ'$  is symmetric unitary,  $(UQ)^*VQ' = ZZ^T$  for some  $Z \in U(n)$  by Lemma 2.1. Set  $W := Z^*(UQ)^* \in U(n)$  so that  $(UQ)^* = ZW$  and  $VQ' = (UQ)ZZ^T = W^*Z^T$ . Thus  $A = W^*(Z^T\Sigma Z)W$ , i.e.,  $A$  is unitarily similar to the complex symmetric matrix  $Z^T\Sigma Z$ .  $\square$

**Corollary 2.3.** Let  $A \in \mathbb{C}_{n \times n}$  be nonsingular, let  $A = V\Sigma U^*$  be a SVD of  $A$ , and let  $\sigma_m$  be the smallest singular value of  $A$ . If  $A$  is unitarily similar to a complex symmetric matrix, then so is  $A_d := V(\Sigma - dI)U^*$  for all  $d \leq \sigma_m$ . If  $A_d$  is unitarily similar to a complex symmetric matrix for some  $d < \sigma_m$ , then so is  $A$ .

*Proof.* Suppose that the nonsingular  $A = V\Sigma U^*$  is unitarily similar to a complex symmetric matrix. Then  $(UQ)^*VQ$  is symmetric for some  $Q \in U(n)$  by Theorem 2.2(3). For any  $d \leq \sigma_m$ ,  $A_d = (VQ)(\Sigma - dI)(UQ)^*$  is a SVD of  $A_d$ . Then the result follows by Theorem 2.2(2).

Suppose that  $A_d$  is unitarily similar to a complex symmetric matrix for some  $d < \sigma_m$ . Since  $A_d$  is nonsingular with SVD  $A_d = V(\Sigma - dI)U^*$ ,  $A = V((\Sigma - dI) + dI)U^*$ , and clearly  $-d \leq 0 < \sigma_m - d$ , the result follows by the same argument as in the previous paragraph.  $\square$

We now give another proof of Theorem 1.1. We first recall the following result in [1, p.5]. See [8] for the symmetric unitary completion problem for a symmetric matrix.

**Lemma 2.4.** A unitary matrix  $U$  is a product of a symmetric unitary matrix (of the form  $e^{iS}$ , where  $S$  is real symmetric) and an orthogonal matrix  $O$ , i.e.,  $U = e^{iS}O$ . It is also true that  $U = O'e^{iS'}$ , where  $O'$  is orthogonal and  $S'$  is real symmetric.

Lemma 2.4 can be used to prove Theorem 1.1. The idea of the proof will be extended to orthogonal symmetric pairs of the compact type in next section.

*Proof of Theorem 1.1.* (1)  $\Rightarrow$  (2). Suppose that  $V^*AV = S$ , where  $V \in U(n)$  and  $S$  is symmetric. By Lemma 2.4,  $V = UO$  with  $U$  symmetric unitary and  $O \in O(n)$ . We have  $A = VSV^* = U(OSO^T)U^*$  and hence  $U^*AU = OSO^T$  is symmetric.

(2)  $\Rightarrow$  (3). Suppose that  $U^*AU = S$  with  $U$  symmetric unitary and  $S$  symmetric. Then  $A = USU^* = (USU^T)(UU^T)^*$  with  $USU^T$  symmetric and  $(UU^T)^*$  symmetric unitary.

(3)  $\Rightarrow$  (4). Suppose that  $A = SU$  with  $S$  symmetric and  $U$  symmetric unitary. Then  $UAU^* = US = U^T S^T = A^T$ .

(4)  $\Rightarrow$  (1). Suppose  $VAV^* = A^T$  for some  $V$  symmetric unitary. By Lemma 2.1, there exists  $U \in U(n)$  such that  $V = U^T U$ . Now  $(U^T U)A(U^T U)^* = A^T$  and hence  $UAU^* = (U^T)^* A^T U^T = (UAU^*)^T$ , which is symmetric.  $\square$

**Remark 2.5.** Theorem 1.1 can be deduced from Theorem 2.2. For example, to show that Theorem 2.2 (2)  $\Rightarrow$  Theorem 1.1 (4), suppose that  $A = Y\Sigma X^*$  is a SVD of  $A$  with  $X^*Y$  symmetric. By Lemma 2.1, we have  $X^*Y = UU^T$  for some  $U \in U(n)$ . Thus  $Y = XU U^T$  and

$$\bar{X}Y^* = (YX^T)^* = (XUU^T X^T)^* = ((XU)(XU)^T)^*.$$

So  $V := \bar{X}Y^*$  is symmetric unitary. Since  $X^*Y = (X^*Y)^T = Y^T \bar{X}$ .

$$\begin{aligned} VAV^* &= (\bar{X}Y^*)(Y\Sigma X^*)(\bar{X}Y^*)^* = \bar{X}\Sigma Y^T \bar{X}(\bar{X})^* \\ &= \bar{X}\Sigma Y^T = (Y\Sigma X^*)^T = A^T. \end{aligned}$$

To show that Theorem 2.2 (2)  $\Rightarrow$  Theorem 1.1 (3), we set  $S := Y\Sigma Y^T$  and  $U := \bar{Y}X^*$ , and note that  $U = \bar{Y}X^* = (\bar{X}Y^*)^T$  is symmetric unitary.

### 3. EXTENSION TO ORTHOGONAL SYMMETRIC LIE ALGEBRAS OF THE COMPACT TYPE

In this section, we extend the result of Vermeer in a form that makes sense in the context of an orthogonal symmetric Lie algebra of the compact type. We will follow the notations in [4].

Let  $\mathfrak{u}$  be a compact semisimple Lie algebra and let  $\theta$  be an involutive automorphism of  $\mathfrak{u}$ . Then

$$\mathfrak{u} = \mathfrak{k}_0 \oplus \mathfrak{p}_*,$$

where  $\mathfrak{k}_0$  and  $\mathfrak{p}_*$  are the +1 and -1 eigenspaces of  $\theta$  respectively. Then

$$[\mathfrak{k}_0, \mathfrak{k}_0] \subset \mathfrak{k}_0, \quad [\mathfrak{k}_0, \mathfrak{p}_*] \subset \mathfrak{p}_*, \quad [\mathfrak{p}_*, \mathfrak{p}_*] \subset \mathfrak{k}_0.$$

Let  $(U, K)$  be a pair associated with  $(\mathfrak{u}, \theta)$ , i.e.,  $U$  is a connected Lie group with Lie algebra  $\mathfrak{u}$  and  $K$  is a (compact) Lie subgroup of  $U$  with Lie algebra  $\mathfrak{k}_0$ . The pair  $(\mathfrak{u}, \theta)$  or  $(U, K)$  is said to be an orthogonal

symmetric pair of the compact type, and  $U/K$  is a Riemannian locally symmetric space. See [4, p.451–455] for classical orthogonal symmetric Lie algebras  $(\mathfrak{u}, \theta)$  of the compact type and their corresponding Riemannian globally symmetric spaces  $U/K$ .

The extension (also denoted by  $\theta$ ) of  $\theta$  to  $\mathfrak{g} := \mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$  defined by

$$\theta(X + iY) = \theta(X) + i\theta(Y), \quad \text{for any } X, Y \in \mathfrak{u}$$

is a (complex) involutive automorphism of  $\mathfrak{g}$ . For any  $u \in U$ ,  $\text{Ad}_U(u)$  is an automorphism of  $\mathfrak{u}$ , which can be extended to  $\mathfrak{g}$  defined by

$$\text{Ad}_U(u)(X + iY) = \text{Ad}_U(u)(X) + i\text{Ad}_U(u)(Y), \quad \text{for any } X, Y \in \mathfrak{u}.$$

The statements of the following theorem are counterparts of Theorem 1.1(1), (2), (4).

**Theorem 3.1.** Let  $(\mathfrak{u}, \theta)$  be an orthogonal symmetric Lie algebra of the compact type. Let  $\mathfrak{u} = \mathfrak{k}_0 \oplus \mathfrak{p}_*$  be the direct decomposition of  $\mathfrak{u}$  into  $\pm 1$ -eigenspaces of  $\theta$ . Let  $(U, K)$  be a pair associated with  $(\mathfrak{u}, \theta)$ . The following statements are equivalent for any  $A \in \mathfrak{g} := \mathfrak{u} \oplus i\mathfrak{u}$ .

- (1)  $\text{Ad}_U(u)A$  is invariant under  $-\theta$  ( $\theta$ , respectively) for some  $u \in U$ .
- (2)  $\text{Ad}_U(u)A$  is invariant under  $-\theta$  ( $\theta$ , respectively) for some  $u \in \exp \mathfrak{p}_*$ .
- (3)  $\text{Ad}_U(u)A = -\theta(A)$  ( $\theta(A)$ , respectively) for some  $u \in \exp \mathfrak{p}_*$ .

*Proof.* We write  $\text{Ad}$  instead of  $\text{Ad}_U$  if there is no ambiguity. Since we work with  $\text{Ad } U = \text{Int } \mathfrak{u}$ , which is the analytic subgroup of  $\text{Aut } \mathfrak{u}$  with Lie algebra  $\text{ad } \mathfrak{u}$  [4, p.126–129],  $\text{Ad } U$  is independent of the choice of  $U$ . In particular we may assume that  $U$  is simply connected. Note that  $U = K \exp \mathfrak{p}_* K$  [4, p.323, Theorem 8.6] and  $\text{Ad}(K)\mathfrak{p}_* = \mathfrak{p}_*$  [4, p.282, Proposition 1.1]. So for each  $u \in U$ , there exist  $k_1, k_2 \in K$  and  $H \in \mathfrak{p}_*$  such that

$$u = k_1(\exp H)k_2 = k_1k_2k_2^{-1}(\exp H)k_2 = (k_1k_2) \exp(\text{Ad}(k_2^{-1})H).$$

Therefore, we have

$$U = K(\exp \mathfrak{p}_*) = (\exp \mathfrak{p}_*)K. \quad (3.1)$$

(1)  $\Rightarrow$  (2). Suppose that  $\text{Ad}(u)A$  is invariant under  $-\theta$  for some  $u \in U$ . By (3.1)  $u = k(\exp H)$  for some  $k \in K$  and  $H \in \mathfrak{p}_*$ , so we have

$$\text{Ad}(u)A = \text{Ad}(k \exp H)A = \text{Ad}(k)(\text{Ad}(\exp H)A).$$

Since  $\text{Ad}(k)\mathfrak{k}_0 = \mathfrak{k}_0$  and  $\text{Ad}(k)\mathfrak{p}_* = \mathfrak{p}_*$  [4, p.282],  $\text{Ad } k$  commutes with  $\theta$  on  $\mathfrak{u}$  and thus on  $\mathfrak{g}$ . Therefore

$$\begin{aligned} \text{Ad}(k)\{-\theta(\text{Ad}(\exp H)A)\} &= -\theta\{\text{Ad}(k)(\text{Ad}(\exp H)A)\} \\ &= \text{Ad}(k)(\text{Ad}(\exp H)A). \end{aligned}$$

Since  $\text{Ad}(k) \in \text{Aut } \mathfrak{g}$ , we have  $-\theta(\text{Ad}(\exp H)A) = \text{Ad}(\exp H)A$ , i.e.,  $\text{Ad}(\exp H)A$  is invariant under  $-\theta$ .

(2)  $\Rightarrow$  (1) is obvious.

(2)  $\Leftrightarrow$  (3). For any  $H \in \mathfrak{p}_*$ , we have

$$\begin{aligned} -\theta\{\text{Ad}(\exp H)A\} &= \theta \text{Ad}(\exp H)\theta^{-1}(-\theta A) \\ &= \theta \exp(\text{ad } H)\theta^{-1}(-\theta A) \\ &= \exp(\theta(\text{ad } H)\theta^{-1})(-\theta A) \\ &= \exp(\text{ad } (\theta H))(-\theta A) \\ &= \text{Ad}(\exp(\theta H))(-\theta A) \\ &= \text{Ad}(\exp(-H))(-\theta A). \end{aligned}$$

Therefore

$$\begin{aligned} -\theta\{\text{Ad}(\exp(H))A\} &= \text{Ad}(\exp(H))A \\ \Leftrightarrow \text{Ad}(\exp(H))A &= \text{Ad}(\exp(-H))(-\theta A) \\ \Leftrightarrow \text{Ad}(\exp(2H))A &= -\theta(A). \end{aligned}$$

Notice that  $-\theta$  can be replaced by  $\theta$  in the above proof.  $\square$

**Remark 3.2.** From the definition of  $\mathfrak{k}_0$  and  $\mathfrak{p}_*$ , the  $-\theta$ -invariant set  $\Omega$  in  $\mathfrak{g} := \mathfrak{u}_{\mathbb{C}}$  is  $(\mathfrak{p}_*)_{\mathbb{C}}$ . Similarly, the  $\theta$ -invariant set in  $\mathfrak{g}$  is  $(\mathfrak{k}_0)_{\mathbb{C}}$ .

We now work out Theorem 3.1 explicitly for the classical orthogonal symmetric Lie algebras  $(\mathfrak{u}, \theta)$  of the compact type.

**Example 3.3.** Type AI ([4, p.451]). Theorem 1.1 essentially corresponds to the orthogonal symmetric pair  $(\mathfrak{u}, \theta)$  with  $\mathfrak{u} = \mathfrak{su}(n)$  and  $\theta(X) = \bar{X}$ , for which we have that  $\mathfrak{k}_0 = \mathfrak{so}(n)$ ,  $\mathfrak{p}_*$  consists of all purely imaginary symmetric traceless matrices,  $U = \text{SU}(n)$ ,  $K = \text{SO}(n)$ , and  $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}(n) \oplus i\mathfrak{su}(n)$ . The extension of  $\theta$  to  $\mathfrak{g}$  is defined by

$$\theta(X+iY) = \theta X + i\theta Y = -(\bar{X})^* - i(\bar{Y})^* = -(X+iY)^T, \quad X, Y \in \mathfrak{su}(n).$$

Thus  $\theta(A) = -A^T$  for all  $A \in \mathfrak{sl}_n(\mathbb{C})$  so that taking the transpose on  $\mathfrak{sl}_n(\mathbb{C})$  amounts to taking  $-\theta$  in Theorem 3.1. Thus the  $-\theta$ -invariant set  $\Omega$  in  $\mathfrak{sl}_n(\mathbb{C})$  consists of all traceless symmetric matrices.

The set  $\exp \mathfrak{p}_*$  consists of all special symmetric unitary matrices. Since  $\text{SU}(n)$  is simply connected, applying [4, p.323, Theorem 8.6] on  $\text{SU}(n)$  ensures that every special unitary matrix is the product of a special orthogonal matrix and a special symmetric unitary matrix; this corresponds to Lemma 2.4.

If we consider  $\theta$  in Theorem 3.1 instead of  $-\theta$ , where  $\theta(A) = -A^T$  for all  $A \in \mathfrak{sl}_n(\mathbb{C})$ , we then have the following result, with an appropriate translation.

**Theorem 3.4.** Let  $A \in \mathbb{C}_{n \times n}$ . The following statements are equivalent.

- (1)  $A$  is unitarily similar to the sum of  $(\operatorname{tr} A/n)I$  and a complex skew symmetric matrix.
- (2) There is a symmetric unitary matrix  $U$  such that  $UAU^*$  is the sum of  $(\operatorname{tr} A/n)I$  and a complex skew symmetric matrix.
- (3) There is a symmetric unitary matrix  $V$  such that  $VAV^* = 2(\operatorname{tr} A/n)I - A^T$ .

**Example 3.5.** Type AII ([4, p.452]). When  $\mathfrak{u} = \mathfrak{su}(2n)$  and  $\theta(X) = J_n \bar{X} J_n^{-1} = -J_n X^T J_n^{-1}$ , where

$$J_n := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

we have

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{sp}(n) = \left\{ \begin{pmatrix} Z_1 & -\bar{Z}_2 \\ Z_2 & \bar{Z}_1 \end{pmatrix} : Z_1 \in \mathfrak{u}(n), Z_2^T = Z_2 \right\} \\ \mathfrak{p}_* &= \left\{ \begin{pmatrix} Z_1 & Z_2 \\ \bar{Z}_2 & \bar{Z}_1 \end{pmatrix} : Z_1 \in \mathfrak{su}(n), Z_2 \in \mathfrak{so}_n(\mathbb{C}) \right\}. \end{aligned}$$

Moreover  $U = \mathrm{SU}(2n)$ ,  $K = \mathrm{Sp}(n)$ ,  $\mathfrak{g} = \mathfrak{sl}_{2n}(\mathbb{C})$ , and

$$-\theta(A) = J_n A^T J_n^{-1}, \quad \text{for all } A \in \mathfrak{g}.$$

Hence the  $-\theta$ -invariant set in  $\mathfrak{sl}_{2n}(\mathbb{C})$  is

$$\Omega := \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & \bar{Z}_1 \end{pmatrix} : Z_2, Z_3 \in \mathfrak{so}_n(\mathbb{C}), \operatorname{tr} Z_1 = 0 \right\}.$$

So Theorem 3.1 asserts that each matrix  $A \in \mathfrak{sl}_n(\mathbb{C})$  is  $\mathrm{SU}(2n)$ -similar to a matrix in  $\Omega$  if and only if  $A$  is  $\exp \mathfrak{p}_*$ -similar to a matrix in  $\Omega$ , or equivalently  $A$  is  $\exp \mathfrak{p}_*$ -similar to  $J_n A^T J_n^{-1}$ .

**Example 3.6.** Type AIII ([4, p.452]). When  $\mathfrak{u} = \mathfrak{su}(p+q)$  and  $\theta(X) = I_{p,q} X I_{p,q}$ , where  $I_{p,q} := (-I_p) \oplus I_q$ , we have

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{su}_{p,q} = \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} : Z_1 \in \mathfrak{u}(p), Z_2 \in \mathfrak{u}(q), \operatorname{tr} Z_1 + \operatorname{tr} Z_2 = 0 \right\}, \\ \mathfrak{p}_* &= \left\{ \begin{pmatrix} 0 & Z \\ -Z^* & 0 \end{pmatrix} : Z \in \mathbb{C}_{p \times q} \right\}. \end{aligned}$$

Moreover,  $U = \mathrm{SU}(p+q)$ ,  $K = S(\mathrm{U}(p) \times \mathrm{U}(q))$ ,  $\mathfrak{g} = \mathfrak{sl}_{p+q}(\mathbb{C})$ , and

$$-\theta(A) = -I_{p,q} A I_{p,q}, \quad \text{for all } A \in \mathfrak{g}.$$

Hence the  $-\theta$ -invariant set in  $\mathfrak{sl}_{p+q}(\mathbb{C})$  is

$$\Omega := \left\{ \begin{pmatrix} 0 & Z_1 \\ Z_2 & 0 \end{pmatrix} : Z_1 \in \mathbb{C}_{p \times q}, Z_2 \in \mathbb{C}_{q \times p} \right\}.$$

So Theorem 3.1 asserts that each matrix  $A \in \mathfrak{sl}_{p+q}(\mathbb{C})$  is  $\mathrm{SU}(p+q)$ -similar to a matrix in  $\Omega$  if and only if  $A$  is  $\exp \mathfrak{p}_*$ -similar to a matrix in  $\Omega$ , or equivalently  $A$  is  $\exp \mathfrak{p}_*$ -similar to  $-I_{p,q} A I_{p,q}$ .

**Example 3.7.** Type BDI ([4, p.453]). When  $\mathfrak{u} = \mathfrak{so}(p+q)$  with  $p \geq q$  and  $\theta(X) = I_{p,q} X I_{p,q}$ , we have

$$\begin{aligned} \mathfrak{k}_0 &= \mathfrak{so}(p) \oplus \mathfrak{so}(q) = \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} : Z_1 \in \mathfrak{so}(p), Z_2 \in \mathfrak{so}(q) \right\}, \\ \mathfrak{p}_* &= \left\{ \begin{pmatrix} 0 & Z \\ -Z^T & 0 \end{pmatrix} : Z \in \mathbb{R}_{p \times q} \right\}. \end{aligned}$$

Moreover,  $U = \mathrm{SO}(p+q)$ ,  $K = \mathrm{SO}(p) \times \mathrm{SO}(q)$ ,  $\mathfrak{g} = \mathfrak{so}(p+q, \mathbb{C})$ , and

$$-\theta(A) = -I_{p,q} A I_{p,q}, \quad \text{for all } A \in \mathfrak{g}.$$

Hence the  $-\theta$ -invariant set in  $\mathfrak{so}(p+q, \mathbb{C})$  is

$$\Omega := \left\{ \begin{pmatrix} 0 & Z \\ -Z^T & 0 \end{pmatrix} : Z \in \mathbb{C}_{p \times q} \right\}.$$

So Theorem 3.1 asserts that each matrix  $A \in \mathfrak{so}(p+q, \mathbb{C})$  is  $\mathrm{SO}(p+q)$ -similar to a matrix in  $\Omega$  if and only if  $A$  is  $\exp \mathfrak{p}_*$ -similar to a matrix in  $\Omega$ , or equivalently  $A$  is  $\exp \mathfrak{p}_*$ -similar to  $-I_{p,q} A I_{p,q}$ .

**Example 3.8.** Type DIII ([4, p.453]). When  $\mathfrak{u} = \mathfrak{so}(2n)$  and  $\theta(X) = J_n X J_n^{-1} = -J_n X J_n$ , we have  $\mathfrak{k}_0 = \mathfrak{so}(2n) \cap \mathfrak{sp}(n) \cong \mathfrak{u}(n)$ ,

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} : Z_1, Z_2 \in \mathfrak{so}(n) \right\}.$$

Moreover,  $U = \mathrm{SO}(2n)$ ,  $K = \mathrm{SO}(2n) \cap \mathrm{Sp}(n) \cong \mathrm{U}(n)$ ,  $\mathfrak{g} = \mathfrak{so}_{2n}(\mathbb{C})$ , and

$$-\theta(A) = -J_n A J_n^{-1} = J_n A J_n, \quad \text{for all } A \in \mathfrak{g}.$$

Hence the  $-\theta$ -invariant set in  $\mathfrak{so}_{2n}(\mathbb{C})$  is

$$\Omega := \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} : Z_1, Z_2 \in \mathfrak{so}_n(\mathbb{C}) \right\}.$$

So Theorem 3.1 asserts that each  $2n \times 2n$  complex skew symmetric matrix  $A \in \mathfrak{so}_{2n}(\mathbb{C})$  is  $\mathrm{SO}(2n)$ -similar to a matrix in  $\Omega$  if and only if  $A$  is  $\exp \mathfrak{p}_*$ -similar to a matrix in  $\Omega$ , or equivalently  $A$  is  $\exp \mathfrak{p}_*$ -similar to  $-J_n A J_n^{-1}$ .

**Example 3.9.** Type CI ([4, p.454]). When  $\mathfrak{u} = \mathfrak{sp}(n)$  and  $\theta(X) = J_n X J_n^{-1}$ , we have  $\mathfrak{k}_0 = \mathfrak{sp}(n) \cap \mathfrak{so}(2n) \cong \mathfrak{u}(n)$ ,

$$\mathfrak{p}_* = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} : Z_1, Z_2 \text{ symmetric, purely imaginary} \right\}.$$

Moreover,  $U = \mathrm{Sp}(n)$ ,  $K = \mathrm{SO}(2n) \cap \mathrm{Sp}(n) \cong \mathrm{U}(n)$ ,

$$\mathfrak{g} = \mathfrak{sp}_n(\mathbb{C}) = \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & -Z_1^T \end{pmatrix} : Z_1 \in \mathbb{C}_{n \times n}, Z_2, Z_3 \text{ symmetric} \right\},$$

$$-\theta(A) = -J_n A J_n^{-1} = J_n A J_n, \quad \text{for all } A \in \mathfrak{g}.$$

Hence the  $-\theta$ -invariant set in  $\mathfrak{sp}_n(\mathbb{C})$  is

$$\Omega := \left\{ \begin{pmatrix} Z_1 & Z_2 \\ Z_2 & -Z_1 \end{pmatrix} : Z_1, Z_2 \in \mathbb{C}_{n \times n} \text{ symmetric} \right\}.$$

So Theorem 3.1 asserts that each complex symplectic matrix  $A \in \mathfrak{sp}_n(\mathbb{C})$  is  $\mathrm{Sp}(n)$ -similar to a matrix in  $\Omega$  if and only if  $A$  is  $\exp \mathfrak{p}_*$ -similar to a matrix in  $\Omega$ , or equivalently  $A$  is  $\exp \mathfrak{p}_*$ -similar to  $-J_n A J_n^{-1}$ .

**Example 3.10.** Type CII ([4, p.454]). When  $\mathfrak{u} = \mathfrak{sp}(p+q)$  and  $\theta(X) = K_{p,q} X K_{p,q}$ , where  $K_{p,q} = I_{p,q} \oplus I_{p,q}$ , we have

$$\begin{aligned} \mathfrak{k}_0 &= \left\{ \begin{pmatrix} X_{11} & 0 & X_{13} & 0 \\ 0 & X_{22} & 0 & X_{24} \\ -\bar{X}_{13} & 0 & \bar{X}_{11} & 0 \\ 0 & -\bar{X}_{24} & 0 & \bar{X}_{22} \end{pmatrix} : \begin{pmatrix} X_{11} & X_{13} \\ -\bar{X}_{13} & \bar{X}_{11} \end{pmatrix} \in \mathfrak{sp}(p), \begin{pmatrix} X_{22} & X_{24} \\ -\bar{X}_{24} & \bar{X}_{22} \end{pmatrix} \in \mathfrak{sp}(q) \right\} \\ \mathfrak{p}_* &= \left\{ \begin{pmatrix} 0 & X_{12} & 0 & X_{14} \\ -X_{12}^* & 0 & X_{14}^T & 0 \\ 0 & -\bar{X}_{14} & 0 & \bar{X}_{12} \\ -X_{14}^* & 0 & -X_{12}^T & 0 \end{pmatrix} : X_{12}, X_{14} \in \mathbb{C}_{p \times q} \right\} \\ K &= \left\{ \begin{pmatrix} U_1 & 0 & -\bar{V}_1 & 0 \\ 0 & U_2 & 0 & -\bar{V}_2 \\ V_1 & 0 & \bar{U}_1 & 0 \\ 0 & V_2 & 0 & \bar{U}_2 \end{pmatrix} : \begin{pmatrix} U_1 & -\bar{V}_1 \\ V_1 & \bar{U}_1 \end{pmatrix} \in \mathrm{Sp}(p), \begin{pmatrix} U_2 & -\bar{V}_2 \\ V_2 & \bar{U}_2 \end{pmatrix} \in \mathrm{Sp}(q) \right\} \end{aligned}$$

so that  $\mathfrak{k}_0 \cong \mathfrak{sp}(p+q)$  and  $K \cong \mathrm{Sp}(p) \times \mathrm{Sp}(q)$ ,  $U = \mathrm{Sp}(p+q)$ ,  $\mathfrak{g} = \mathfrak{sp}_{p+q}(\mathbb{C})$ , and

$$-\theta(A) = -K_{p,q} A K_{p,q} \quad \text{for all } A \in \mathfrak{g}.$$

Hence the  $-\theta$ -invariant set in  $\mathfrak{sp}_{p+q}(\mathbb{C})$  is

$$\Omega := \left\{ \begin{pmatrix} 0 & X_{12} & 0 & X_{14} \\ X_{21} & 0 & X_{14}^T & 0 \\ 0 & X_{41}^T & 0 & -X_{21}^T \\ X_{41} & 0 & -X_{12}^T & 0 \end{pmatrix} : X_{12}, X_{14} \in \mathbb{C}_{p \times q}, X_{21}, X_{41} \in \mathbb{C}_{q \times p} \right\}$$

So Theorem 3.1 asserts that each complex symplectic matrix  $A \in \mathfrak{sp}_{p+q}(\mathbb{C})$  is  $\mathrm{Sp}(p+q)$ -similar to a matrix in  $\Omega$  if and only if  $A$  is  $\exp \mathfrak{p}_*$ -similar to a matrix in  $\Omega$ , or equivalently  $A$  is  $\exp \mathfrak{p}_*$ -similar to  $-K_{p,q}AK_{p,q}$ .

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