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GENERALIZATON OF KY FAN-AMIR-MOÉZ-HORN-MIRSKY'S RESULT ON THE EIGENVALUES AND REAL SINGULAR VALUES OF A MATRIX

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ABSTRACT. Ky Fan's result states that the real parts of the eigenvalues of an $n \times n$ complex matrix x are majorized by the eigenvalues of the Hermitian part of x. The converse was established by Amir-Moéz and Horn, and Mirsky, independently. We generalize the results in the context of complex semisimple Lie algebra. The real case is also discussed.

1. Introduction

A result of Ky Fan [6] asserts that the real parts of the eigenvalues, denoted by $\alpha \in \mathbb{R}^n$, of each $n \times n$ complex matrix x are majorized [15, p.239] by the eigenvalues $\beta \in \mathbb{R}^n$ of the Hermitian part $\frac{1}{2}(x + x^*)$ of x. This amounts to $\sum_{i=1}^k \alpha_i \le \sum_{i=1}^k \beta_i$ for all $k = 1, \dots, n-1$, and $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i$ after rearranging the entries of α and β, respectively, in nonincreasing order. The converse was established by Amir-Moéz and Horn [1], and independently by Mirsky [17]. It was later rediscovered by Sherman and Thompson [20]. That is, if $\gamma \in \mathbb{C}^n$ and $\beta \in \mathbb{R}^n$ such that the real part of γ is majorized by β, then there exists a complex $n \times n$ matrix x such that γ are eigenvalues of x and β are the eigenvalues of the Hermitian part of x. The results are valid for the imaginary part of the eigenvalues of x and $\frac{1}{2}(x - x^*)$. The study can be traced back to some old results of Bendixson [3], Hirsch [9], and Bromwich [4]. Also see [15, p.237-239].

With an appropriate translation of x we may view x as an element of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ of the special linear group $G = \mathrm{SL}(n, \mathbf{C})$. The special unitary group $K = \mathrm{SU}(n)$ is a maximal compact subgroup of G. The diagonal matrices in $\mathfrak{g}$ form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and those with purely imaginary diagonal entries form a Cartan subalgebra $\mathfrak{t}$ of the Lie algebra $\mathfrak{k} = \mathfrak{su}(n)$ of K. As a real K-module, $\mathfrak{g}$ is just the direct sum of two copies of the adjoint module $\mathfrak{k}$ of K: $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$ which in our case is the well-known Hermitian decomposition of a complex matrix. By the Schur Triangularization Theorem for complex matrices, the eigenvalues of x may be viewed as the image of an element $y \in K \cdot x \cap \mathfrak{b}$ under the orthogonal projection $\mathfrak{g} \to \mathfrak{h}$ with respect to the Killing form, where K acts on $\mathfrak{g}$ via the restriction of the adjoint representation, $K \cdot x$ is the orbit of x under the action of K, and $\mathfrak{b}$ is the Borel subalgebra consisting of $n \times n$ upper triangular matrices. Thus taking the real part of the eigenvalues of x amounts to sending y via the projection $\pi : \mathfrak{g} \to i\mathfrak{t}$ to its image. The majorization relation $\alpha \prec \beta$ is equivalent to $\alpha \in \operatorname{conv} S_n \beta$ for

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$\alpha, \beta \in \mathbb{R}^n$ [2, 15], a well known result of Hardy, Littlewood and Pólya. So the result of Ky Fan may be stated as $\pi(K \cdot (x+z) \cap \mathfrak{b}) \subset \operatorname{conv} Wz$, and the result of Amir-Moéz-Horn and Mirsky may be written as $\operatorname{conv} Wz \subset \cup_{x\in\mathfrak{k}} \pi(K \cdot (x+z) \cap \mathfrak{b})$, where $z \in i\mathfrak{t}$ and W is the Weyl group of $(\mathfrak{g}, \mathfrak{h})$ ($W = S_n$ in this situation). They may then be combined as

$$\cup_{x\in\mathfrak{k}} \pi(K \cdot (x+z) \cap \mathfrak{b}) = \operatorname{conv} Wz,$$

or $\pi((\mathfrak{k} + K \cdot z) \cap \mathfrak{b}) = \operatorname{conv} Wz$. In this paper we prove that the statement is true for any complex semisimple Lie algebra $\mathfrak{g}$. We also discuss the real case.

2. Preliminaries

Let K be a connected compact semisimple Lie group, G its complexification, and let $\mathfrak{k}$ and $\mathfrak{g}$ be their respective Lie algebras. Thus $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. We fix a maximal torus T of K and denote its Lie algebra by $\mathfrak{t}$. Then $\mathfrak{h} = \mathfrak{t} \oplus i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$.

Let R be the root system of $(\mathfrak{g}, \mathfrak{h})$ and fix a base Π for R. The set of positive roots (with respect to Π) is denoted by R^+. Let $\mathfrak{g}^\alpha$ denote the root space of a root α. We introduce the maximal nilpotent subalgebras $\mathfrak{n}$ and $\mathfrak{n}^-$ of $\mathfrak{g}$:

$$\mathfrak{n} = \dot{\sum}_{\alpha\in R^+} \mathfrak{g}^\alpha, \quad \mathfrak{n}^- = \dot{\sum}_{\alpha\in R^+} \mathfrak{g}^{-\alpha}.$$

Then $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ is a Borel subalgebra of $\mathfrak{g}$. The coroot corresponding to a root α is denoted by H_α. Recall that $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ is a 1-dimensional subspace of $\mathfrak{h}$ and H_α is the unique element of $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ such that $\alpha(H_\alpha) = 2$. The Weyl group of $(\mathfrak{g}, \mathfrak{h})$ will be denoted by W.

We denote by θ the Cartan involution of $\mathfrak{g}$ (when viewed as a real Lie algebra): It is identity on $\mathfrak{k}$ and negative identity on $i\mathfrak{k}$. We remark that $\theta(\mathfrak{h}) = \mathfrak{h}$ and $\theta(\mathfrak{g}^\alpha) = \mathfrak{g}^{-\alpha}$ for all $\alpha \in R$.

The Killing form of $\mathfrak{g}$ will be denoted by φ. Unless stated otherwise, the orthogonal complements will be taken with respect to φ.

3. The complex semisimple case

In this section we assume that $\mathfrak{g}$ is a complex semisimple Lie algebra and use the notations in the previous section.

Proposition 3.1. [5] *$K \cdot x$ intersects $\mathfrak{b}$ for each $x \in \mathfrak{g}$.*

The following is a generalization of Ky Fan-Amir-Moéz-Horn-Mirsky's result.

Theorem 3.2. *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\pi : \mathfrak{g} \to i\mathfrak{t}$ be the orthogonal projection with respect to the Killing form. If $z \in i\mathfrak{t}$, then*

$$\cup_{x\in\mathfrak{k}} \pi(K \cdot (x+z) \cap \mathfrak{b}) = \mathit{conv}\, Wz,$$

or equivalently, $\pi((\mathfrak{k} + K \cdot z) \cap \mathfrak{b}) = \mathit{conv}\, Wz$. In particular, for each $w \in \mathfrak{g}$, $\pi(K \cdot w \cap \mathfrak{b}) \subset \mathit{conv}\, Wz$, where $z \in K \cdot \frac{1}{2}(w - \theta w) \cap i\mathfrak{t}$.

Proof. Given $x \in \mathfrak{k}$, since $K \cdot x \subset \mathfrak{k}$ and $K \cdot z \subset i\mathfrak{k}$, it follows

$$\pi(K \cdot (x+z) \cap \mathfrak{b}) \subset \pi(K \cdot (x+z)) = \pi(K \cdot z) = \operatorname{conv} Wz,$$

by Kostant's result [14, Theorem 8.2]. Hence $\cup_{x\in\mathfrak{k}} \pi(K \cdot (x+z) \cap \mathfrak{b}) \subset \cup_{x\in\mathfrak{k}} \pi(K \cdot (x+z)) \subset \operatorname{conv} Wz$.

Conversely let $\beta \in \operatorname{conv} Wz$. By Kostant's result again, there exists $y \in K\cdot z$ such that $\pi(y) = \beta$. Recall the root space decomposition [12]: $\mathfrak{g} = \mathfrak{h}\dot{+}\sum_{\alpha\in R^+}(\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha})$. The direct sum $\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}$ may not be orthogonal. Let $y = y_0 + \sum_{\alpha\in R^+}(y_\alpha + y_{-\alpha})$, where $y_0 \in \mathfrak{h}$, $y_\alpha \in \mathfrak{g}^\alpha$ and $y_{-\alpha} \in \mathfrak{g}^{-\alpha}$. Since $y \in i\mathfrak{k}$, $i\mathfrak{k}$ is the -1 eigenspace of θ, and $\theta\mathfrak{g}^\alpha = \mathfrak{g}^{-\alpha}$ ($\alpha \neq 0$), we have

$$-y_0 + \sum_{\alpha\in R^+}(-y_\alpha - y_{-\alpha}) = -y = \theta y = \theta y_0 + \sum_{\alpha\in R^+}(\theta y_\alpha + \theta y_{-\alpha}).$$

Since the sums are direct, $y_0 \in i\mathfrak{t} \subset i\mathfrak{k}$ and $y_{-\alpha} = -\theta y_\alpha$ for all $\alpha \in R$. Then $y = y_0 + \sum_{\alpha\in R^+}(y_\alpha - \theta y_\alpha)$, and $y_0 = \pi(y) = \beta$. Set $x := \sum_{\alpha\in R^+}(y_\alpha + \theta y_\alpha) \in \mathfrak{k}$. Then $x + y = y_0 + 2\sum_{\alpha\in R^+} y_\alpha \in (x + K\cdot z)\cap\mathfrak{b}$. Clearly $\pi(x+y) = \pi(y) = \beta$.

Now

$$\begin{aligned}
\cup_{x\in\mathfrak{k}}\,\pi(K\cdot(x+z)\cap\mathfrak{b}) &= \pi(\cup_{x\in\mathfrak{k}}\,K\cdot(x+z)\cap\mathfrak{b})\\
&= \pi(\cup_{x\in\mathfrak{k}}\cup_{k\in K}\ (k\cdot x + k\cdot z)\cap\mathfrak{b})\\
&= \pi(\cup_{k\in K}\cup_{x\in\mathfrak{k}}\ (k\cdot x + k\cdot z)\cap\mathfrak{b}))\\
&= \pi(\cup_{k\in K}\,(\mathfrak{k} + k\cdot z)\cap\mathfrak{b}))\\
&= \pi((\mathfrak{k} + K\cdot z)\cap\mathfrak{b}).
\end{aligned}$$

For $w \in \mathfrak{g}$, $K\cdot w\cap\mathfrak{b}$ is nonempty by Proposition 3.1. We decompose it as $w = \frac{1}{2}(w+\theta w) + \frac{1}{2}(w-\theta w)$. Clearly $\pi(K\cdot w\cap\mathfrak{b}) \subset \pi_{x\in\mathfrak{k}}\pi(K\cdot(\frac{1}{2}(w+\theta w) + \frac{1}{2}(w-\theta w))\cap\mathfrak{b}) \subset \operatorname{conv} Wz$ where $z \in K\cdot\frac{1}{2}(w-\theta w)\cap i\mathfrak{t}$. $\square$

Remark 3.3. When $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, the theorem is simply Ky Fan-Amir-Moéz-Horn-Mirsky's result with an appropriate translation. Amir-Moéz and Horn [1] introduced the term *real singular values* (*imaginary singular values*, respectively) for the eigenvalues of $\frac{1}{2}(x + x^*)$ ($\frac{1}{2i}(x - x^*)$), where x^* is the complex conjugate of $x \in \mathfrak{g}$.

Remark 3.4. The statement of Theorem 3.2 remains true when the Cartan subspace $i\mathfrak{k}$ is replaced by $\mathfrak{k}$. Since $\mathfrak{g}$ is an inner product space equipped with the natural inner product $(x,y) = -\varphi(x,\theta y)$, and $\mathfrak{k}$ and $i\mathfrak{k}$ are orthogonal,

$$\|x\|^2 = \|\frac{1}{2}(x+\theta x)\|^2 + \|\frac{1}{2}(x-\theta x)\|^2.$$

When $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$, it simply asserts that the square of the Frobenius norm of x is the sum of squares of the real and imaginary singular values of x [1, Theorem 5].

Remark 3.5. Sherman and Thompson [20] states the converse of Ky Fan's result slightly different: If z is a given Hermitian matrix with eigenvalues $\beta \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}^n$, satisfying $\alpha \prec \beta$, then there exists a skew Hermitian matrix x such that α is the real part of the eigenvalues of $x + z$. It is indeed equivalent to Amir-Moéz-Mirsky's result: there exists an $n \times n$ complex matrix y such that the real part of the eigenvalues of y are α and the eigenvalues of $\frac{1}{2}(y + y^*)$ are β. It is because that the eigenvalues of y are invariant under conjugation, and the spectral theorem implies that $k(\frac{1}{2}(y+y^*))k^{-1} = z$ for some $U \in SU(n)$.

Example 3.6. [12, p.85] Consider the simple complex Lie algebra $\mathfrak{c}_n$ which is realized as $\mathfrak{g} = \mathfrak{sp}(n,\mathbb{C}) = \mathfrak{sp}(n) + i\mathfrak{sp}(n)$, and $K = Sp(n)$, symplectic group [12] which consists of the matrices of the form

$$\begin{pmatrix} U & -\overline{V} \\ V & \overline{U} \end{pmatrix} \in U(2n).$$

Now

$$\mathfrak{h} = \{\mathrm{diag}\,(h_1, \dots, h_n, -h_1, \dots, -h_n) : h_1, \dots, h_n \in \mathbb{C}\},$$

which will be identified with $\mathbb{C}^n$ and $i\mathfrak{t}$ with $\mathbb{R}^n$ in the natural way. The positive roots are

$$\{e_i \pm e_j,\, 2e_k : 1 \le i < j \le n, 1 \le k \le n\}.$$

The corresponding root spaces are

$$\begin{aligned} \mathfrak{g}^{e_i - e_j} &= \mathbb{C}(E_{i,j} - E_{j+n,i+n}), \\ \mathfrak{g}^{e_i + e_j} &= \mathbb{C}(E_{i,j+n} + E_{j,i+n}), \\ \mathfrak{g}^{2e_k} &= \mathbb{C}(E_{k,k+n}), \end{aligned}$$

where E_{ij} denotes the matrix with (i,j) entry 1 and zero otherwise. The Weyl group W of $(\mathfrak{g}, \mathfrak{h})$ acts on $\mathfrak{h}$ (viewed as a real vector space) and thus on $i\mathfrak{t}$:

$$(h_1, \dots, h_n) \mapsto (\pm h_{\sigma(1)}, \dots, \pm h_{\sigma(n)}), \quad \sigma \in S_n,$$

where S_n is the symmetric group. The simple roots Δ [11, p.64] are

$$\alpha_i = e_i - e_{i+1},\ i = 1, \dots, n-1, \quad \alpha_n = 2e_n,$$

and the fundamental dominant weights [11, p.67] (also known as the dual basis of Δ) are

$$\lambda_i = \sum_{k=1}^{i} e_k, \quad i = 1, \dots, n.$$

The (closed) fundamental Weyl chamber (the cone generated by the fundamental dominant weights) $i\mathfrak{t}_+$ is

$$i\mathfrak{t}_+ := \{(h_1, \dots, h_n) : h_1 \ge \cdots \ge h_n \ge 0\}.$$

The dual cone of $i\mathfrak{t}_+$ in $i\mathfrak{t}$ (the cone generated by the simple roots), defined as

$$\mathrm{dual}_{\,i\mathfrak{t}}\, i\mathfrak{t}_+ := \{x \in i\mathfrak{t} : (x, u) \ge 0,\ \text{for all } u \in i\mathfrak{t}_+\},$$

may be written as

$$\mathrm{dual}_{\,i\mathfrak{t}}\, i\mathfrak{t}_+ = \{x \in i\mathfrak{t} : \sum_{k=1}^{j} x_k \ge 0,\ j = 1, \dots, n\}.$$

Recall that $\alpha \in \mathrm{conv}\, W\beta$ if and only if $\beta - \alpha \in \mathrm{dual}_{\,i\mathfrak{t}}\, i\mathfrak{t}_+$, provided that $\alpha, \beta \in i\mathfrak{t}_+$ [14, Lemma 3.3]. Thus the n absolute values of the real parts of $(h_1, \dots, h_n)$, where $(h_1, \dots, h_n, -h_1, \dots, -h_n) \in \pi(K \cdot x \cap \mathfrak{b})$, (notice that $\pm h_j$ are not the eigenvalues of x in general) are weakly majorized [2] by the n nonnegative eigenvalues of $\frac{1}{2}(x + x^*)$. The converse is true.

Example 3.7. [12, p.85] Consider the simple complex Lie algebra $\mathfrak{d}_n$ which is realized as $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}) = \mathfrak{so}(2n) + i\mathfrak{so}(2n)$, the algebra of $2n \times 2n$ complex skew symmetric matrices, and $K = SO(2n)$. Now

$$\mathfrak{h} = \{\begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & h_n \\ -h_n & 0 \end{pmatrix} : h_1, \dots, h_n \in \mathbb{C}\},$$

which will be identified with $\mathbb{C}^n$ and $i\mathfrak{t}$ with $\mathbb{R}^n$ in the natural way. Similarly,

$$\mathfrak{t} = \{\begin{pmatrix} 0 & t_1 \\ -t_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & t_n \\ -t_n & 0 \end{pmatrix} : t_1, \dots, t_n \in \mathbb{R}\}.$$

The positive roots are $\{e_i \pm e_j : 1 \le i < j \le n\}$. The corresponding root spaces are

$$\begin{aligned} \mathfrak{g}^{e_i-e_j} &= \mathbb{C}\begin{pmatrix} 0 & X_{ij} \\ -X_{ij}^T & 0 \end{pmatrix}, \\ \mathfrak{g}^{e_i+e_j} &= \mathbb{C}\begin{pmatrix} 0 & Y_{ij} \\ -Y_{ij}^T & 0 \end{pmatrix}, \end{aligned}$$

where

$$X_{ij} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}, \quad Y_{ij} = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$$

are the (i,j) blocks (2×2 matrices) of the indicated matrices.

The Weyl group W of $(\mathfrak{g},\mathfrak{h})$ acts on $\mathfrak{h}$ (viewed as a real vector space) and thus on $i\mathfrak{t}$:

$$(h_1,\ldots,h_n) \mapsto (\pm h_{\sigma(1)},\ldots,\pm h_{\sigma(n)}), \quad \sigma \in S_n, \text{ number of negative signs is even.}$$

The simple roots Δ [11, p.64] are

$$\alpha_i = e_i - e_{i+1},\ i=1,\ldots,n-2, \quad \alpha_{n-1} = e_{n-1}-e_n, \quad \alpha_n = e_{n-1}+e_n,$$

and the fundamental dominant weights are

$$\lambda_i = \sum_{k=1}^{i} e_k,\ i=1,\ldots,n-2, \quad \lambda_{n-1} = \frac{1}{2}(\sum_{k=1}^{n-1} e_k - e_n),\ \lambda_n = \frac{1}{2}\sum_{k=1}^{n} e_k.$$

The (closed) fundamental Weyl chamber $i\mathfrak{t}_+$ is

$$i\mathfrak{t}_+ := \{(h_1,\ldots,h_n) : h_1 \ge \cdots \ge h_{n-1} \ge |h_n|\}.$$

The dual cone of $i\mathfrak{t}_+$ in $i\mathfrak{t}$ is

$$\mathrm{dual}_{\,i\mathfrak{t}} i\mathfrak{t}_+ = \{x \in i\mathfrak{t} : \sum_{k=1}^{j} x_k \ge 0, \sum_{k=1}^{n-1} x_k - x_n \ge 0,\ j=1,\ldots,n\}.$$

The set $K\cdot x \cap \mathfrak{b}$ does not provide, in general, the eigenvalues of $x \in \mathfrak{g}$, though the eigenvalues of x occur in pair but opposite in sign (need not be real). The eigenvalues of the skew symmetric matrix $\frac{1}{2}(x-\overline{x})$ (also Hermitian) are $\pm\beta_j$, $\beta_j \in \mathbb{R}$, $j=1,\ldots,n$. Arrange β's so that $\beta_1 \ge \cdots \ge \beta_n$. The real parts of $(h_1,\ldots,h_n)$, where $h = (h_1,\ldots,h_n,-h_1,\ldots,-h_n) \in \pi(K\cdot x\cap\mathfrak{b})$, are denoted by $\alpha_1,\ldots,\alpha_n$ such that $|\alpha_1| \ge \cdots \ge |\alpha_n|$. Let r be the number of negative entries among α_i, $i=1,\ldots,n$. Then [24, 23]

$$\begin{aligned} \sum_{i=1}^{k} |\alpha_i| &\le \sum_{i=1}^{k} \beta_i, \quad k=1,\ldots,n-1, \\ \sum_{i=1}^{n-1} |\alpha_i| - |\alpha_n| &\le \sum_{i=1}^{n-1} \beta_i - \beta_n, \\ \sum_{i=1}^{n-1} |\alpha_i| - (-1)^r [\mathrm{sign}\,\mathrm{Pf}(\frac{1}{2}(x+x^t))]\,|\alpha_n| &\le \sum_{i=1}^{n-1} \beta_i(A) - \beta_n, \end{aligned}$$

where $\mathrm{Pf}\,(\frac{1}{2}(x-\overline{x}))$ denotes the Pfaffian [7, Appendix D] of the $2n\times 2n$ skew symmetric matrix $\frac{1}{2}(x-\overline{x})$.

4. The eigenvalues and the real and imaginary singular values for $\mathfrak{sl}(2,\mathbb{C})$ and $\mathfrak{sl}(2,\mathbb{R})$

More restriction on the eigenvalues of $x \in \mathfrak{sl}(n,\mathbb{C})$ is expected, if the real *and* imaginary singular values of $x \in \mathfrak{sl}(n,\mathbb{C})$ are known. The following is the simplest case and it shows that the norm condition in Remark 3.4 is not sufficient.

Proposition 4.1. *Let $\alpha, \beta \in \mathbb{R}$ and $a+ib \in \mathbb{C}$. Then there exists $x \in \mathfrak{sl}(2,\mathbb{C})$ whose eigenvalues, real singular values, and imaginary singular values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively, if and only if $(-a,a) \prec (-\alpha,\alpha)$, $(-b,b) \prec (-\beta,\beta)$, and $\beta^2 - b^2 = \alpha^2 - a^2$.*

Proof. Let $x \in \mathfrak{sl}(2,\mathbb{C})$ whose eigenvalues, real singular values, and imaginary singular values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively. After an appropriate unitary similarity, we may assume that x is in upper triangular form:

$$x = \begin{pmatrix} a+ib & c \\ 0 & -a-ib \end{pmatrix}, (x+x^*)/2 = \begin{pmatrix} a & c/2 \\ \overline{c}/2 & -a \end{pmatrix}, (x-x^*)/2i = \begin{pmatrix} b & c/2i \\ -\overline{c}/2i & -b \end{pmatrix}.$$

The eigenvalues of the matrices are $\pm(a+ib)$, $\pm\alpha = \pm(a^2 + \frac{1}{4}|c|^2)^{1/2}$, and $\pm\beta = \pm(b^2 + \frac{1}{4}|c|^2)^{1/2}$. So $(-a,a) \prec (-\alpha,\alpha)$, $(-b,b) \prec (-\beta,\beta)$ and $\beta^2 - b^2 = \alpha^2 - a^2 = \frac{1}{4}|c|^2$. Conversely, if the conditions are satisfied, the above triangular matrix x (thus not unique) is the required one with $\alpha^2 - a^2 = \frac{1}{4}|c|^2$, □

Proposition 4.2. *Let $\alpha, \beta \in \mathbb{R}$ and $a+ib \in \mathbb{C}$. Then there exists $x \in \mathfrak{sl}(2,\mathbb{R})$ whose eigenvalues, real singular values, and imaginary singular values are $\pm(a+ib)$, $\pm\alpha$, and $\pm\beta$, respectively, if and only if (1) $b = 0$, $(-a,a) \prec (-\alpha,\alpha)$, and $\beta^2 = \alpha^2 - a^2$, or (2) $a = \alpha = 0$, $b = \pm\beta$.*

Proof. If the eigenvalues of $x \in \mathfrak{sl}(2,\mathbb{R})$ are complex, they must be conjugate to each other, that is, $\pm ib$, $b \in \mathbb{R}$. Otherwise, they must be of the form $\pm a$, $a \in \mathbb{R}$. By Proposition 4.1, or by observing each $x \in \mathfrak{sl}(2,\mathbb{R})$ is (special) orthogonally similar to one of the forms:

$$(a) \begin{pmatrix} a & c \\ 0 & -a \end{pmatrix}, \quad a, c \in \mathbb{R}, \qquad (b) \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}, \quad b \in \mathbb{R},$$

accordingly, we have the necessary conditions. Conversely, (1) let x be in the form (a) whose eigenvalues, real singular values, and imaginary singular values are $\pm a$, $\pm\alpha = \pm(a^2 + \frac{1}{4}c^2)^{1/2}$, and $\pm\beta = \pm(\frac{1}{4}c^2)^{1/2}$. Thus set $c = \pm 2|\beta|$; (2) let x be in the form (b) and it is obvious. □

5. The real semisimple case

The proof [1] given by Amir-Moéz and Horn for the converse of Ky Fan's result also works for $\mathfrak{sl}(n,\mathbb{R})$, a *normal* real form of $\mathfrak{sl}(n,\mathbb{C})$. In the complex semisimple case $\mathfrak{g}$, all maximal solvable subalgebras in $\mathfrak{g}$ are conjugate via the adjoint group of $\mathfrak{g}$ [11, Section 16.4] (it is also true for Cartan subalgebras). The Borel subalgebra $\mathfrak{b}$ in Section 2 and 3 is the "standard" one with respect to the chosen Cartan subalgebra $\mathfrak{h} = \mathfrak{t} + i\mathfrak{t}$ and the basis Π for the root system R. However, in the real case, there are different conjugacy classes of maximal solvable subalgebras [16, 18, 19]. The conjugacy classes may be obtained via the nonconjugate Cartan subalgebras [16]. It is well known [13, 22] that Cartan subalgebras of a real semisimple Lie algebra are not conjugate in general.

From now on in this section we denote by $\mathfrak{g}$ a real semisimple Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated with the Cartan involution θ, that is, $\mathfrak{k}$ is the $+1$ eigenspace of θ and $\mathfrak{p}$ is the -1 eigenspace of θ. Fix a maximal abelian subalgebra $\mathfrak{a}_{\mathfrak{p}}$ in $\mathfrak{p}$. Let

$$\mathfrak{g} = \mathfrak{g}^0 \dot{+} \dot{\sum}_{\alpha\in R^+} (\mathfrak{g}^\alpha + \mathfrak{g}^{-\alpha}),$$

be the *restricted root-space decomposition* of $\mathfrak{g}$ relative to $\mathfrak{a}_{\mathfrak{p}}$ [12, p.313], where R^+ is the set of positive roots (with respect to a fixed base Π of R) of the root system R of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$, and

$$\mathfrak{g}^\alpha = \{X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{a}_{\mathfrak{p}}\}, \qquad \alpha \in R.$$

We also have the orthogonal sum

$$\mathfrak{g}^0 = \mathfrak{a}_{\mathfrak{p}} \dot{+} \mathfrak{m},$$

where $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a}_{\mathfrak{p}})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. Notice that [12, p.313]

$$\mathfrak{k} \cap \mathfrak{g}^0 = \mathfrak{m}, \qquad \mathfrak{p} \cap \mathfrak{g}^0 = \mathfrak{a}_{\mathfrak{p}}.$$

The Weyl group of $(\mathfrak{g}, \mathfrak{a}_{\mathfrak{p}})$ will be denoted by W. Let $\mathfrak{b} := \mathfrak{a}_{\mathfrak{p}} + \dot{\sum}_{\alpha\in R^+} \mathfrak{g}^\alpha$.

Example 5.1. [12, p.313-314] When $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{F})$ where $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, $\mathfrak{k} \subset \mathfrak{g}$ consists of the skew Hermitian matrices of $\mathfrak{g}$ and $\mathfrak{p} \subset \mathfrak{g}$ consists of the Hermitian matrices, the maximal abelian subspace $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{p}$ is the algebra of traceless real diagonal matrices, and $\mathfrak{m}$ consists of all skew Hermitian diagonal matrices in $\mathfrak{g}$. Thus for $\mathbb{F} = \mathbb{R}$, $\mathfrak{m} = 0$; for $\mathbb{F} = \mathbb{C}$, $\mathfrak{m}$ consists of all traceless diagonal pure imaginary matrices. For $\mathbb{F} = \mathbb{H}$, $\mathfrak{m}$ consists of all diagonal matrices whose diagonal entries x_j satisfies $\overline{x}_j = -x_j$. The restricted root space $\mathfrak{g}^{e_i - e_j}$ consists of all matrices with (i, j) entry 1 and zero otherwise, $1 \le i \ne j \le n$, where e_i denotes the functional on $\mathfrak{a}_{\mathfrak{p}}$ evaluating the ith diagonal entry of each element of $\mathfrak{a}_{\mathfrak{p}}$.

The proof of the following is similar to that of Theorem 3.2 and is omitted.

Theorem 5.2. *Let $\mathfrak{g}$ be a real semisimple Lie algebra $\mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{a}_{\mathfrak{p}}$ be the orthogonal projection with respect to the Killing form. Then for each $\beta \in \mathfrak{a}_{\mathfrak{p}}$,*

$$\pi((\mathfrak{k} + K \cdot \beta) \cap \mathfrak{b})) = conv\, W\beta.$$

Remark 5.3. Theorem 5.2 provides Amir-Moéz-Horn't type result for the *real* semisimple Lie algebras $\mathfrak{sl}(n, \mathbb{R})$. We can obtain Theorem 3.2 from Theorem 5.2: If $\mathfrak{g}_1$ is a complex semisimple Lie algebra, $\mathfrak{g} = \mathfrak{g}_1^{\mathbb{R}}$ its realification, let $\mathfrak{k}$ be a compact real form of $\mathfrak{g}_1$. Then $\mathfrak{g} = \mathfrak{k} + i\mathfrak{k}$ is a Cartan decomposition of $\mathfrak{g}$. Taking $\mathfrak{a}_{\mathfrak{p}} = i\mathfrak{t}$ where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$, $\mathfrak{a} = \mathfrak{t} + i\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ and is the realification of a Cartan subalgebra $\mathfrak{h}_1$ of $\mathfrak{g}_1$. The root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{a}_p$ is the same as the restricted root space decomposition of $\mathfrak{g}_1$ with respect to $\mathfrak{h}_1$.

Remark 5.4. The algebra $\mathfrak{s} = \mathfrak{g}^0 + \dot{\sum}_{\alpha\in R^+} \mathfrak{g}^\alpha$ is called the *standard maximal solvable subalgebra* of $\mathfrak{g}$ which contains the *maximally vector Cartan subalgebra* $\mathfrak{g}^0$ [21, p.405], with respect to $\mathfrak{a}_{\mathfrak{p}}$ and Π. Clearly $\mathfrak{b} := \mathfrak{a}_{\mathfrak{p}} + \dot{\sum}_{\alpha\in R^+} \mathfrak{g}^\alpha \subset \mathfrak{s}$.

We now begin our discussion of the real counterpart of Proposition 3.1 and Theorem 5.5. It is well known [10] that for any $x \in \mathfrak{sl}(n, \mathbb{R})$, there exists $k \in SO(n)$

such that kxk^{-1} is of block upper triangular form where the (main diagonal) blocks are either 1×1 or 2×2:

$$\begin{pmatrix} A_1 & \dots & \dots & \dots & \dots & & \dots \\ & A_2 & \dots & \dots & \dots & & \dots \\ & & \ddots & & \dots & & \dots \\ & & & A_j & \dots & & \\ & & & & a_{2j+1} & & \dots \\ & & & & & \ddots & \dots \\ & & & & & & a_n \end{pmatrix},$$

with zero trace, where $A_k = \begin{pmatrix} a_k & b_k \\ -b_k & a_k \end{pmatrix}$, $k = 1, \dots, j$. Indeed the above forms are associated with the maximal solvable subalgebras of $\mathfrak{sl}(n, \mathbb{R})$. There are

$$N_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right]$$

(the Fibonacci number defined by $N_n = N_{n-1} + N_{n-2}$, $N_1 = 1$, $N_2 = 2$) conjugacy classes of maximal solvable subalgebras [18] of $\mathfrak{sl}(n, \mathbb{R})$.

Certainly we $K \cdot x$ may not interest $\mathfrak{b} := \mathfrak{a}_\mathfrak{p} + \sum_{\alpha \in R^+} \mathfrak{g}^\alpha$ for some $x \in \mathfrak{g}$ in view of $\mathfrak{sl}(n, \mathbb{R})$ in which $\mathfrak{b}$ may be viewed as the algebra of real upper triangular matrices. Motivated by Proposition 3.1 and the case $\mathfrak{sl}(n, \mathbb{R})$, we now ask whether for any element x in the real semisimple Lie algebra $\mathfrak{g}$, $K \cdot x$ intersects some maximal solvable subalgebra $\mathfrak{s}$. To be specific, we recall some basic notions. Fix a Cartan decomposition of the real semisimple Lie algebra $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let θ be the associated Cartan involution. Fix a maximal abelian subalgebra $\mathfrak{a}_\mathfrak{p} \subset \mathfrak{p}$ in $\mathfrak{p}$. Let $\mathfrak{a}$ be the Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_\mathfrak{p}$, that is, $\mathfrak{a} := \mathfrak{g}^0$. A θ-stable Cartan subalgebra (and will simply be called Cartan subalgebra) of $\mathfrak{g}$ is a subalgebra $\mathfrak{c}$ that is maximal among abelian θ-stable subalgebras of $\mathfrak{g}$. The Cartan subalgebras of $\mathfrak{g}$ have the same dimension which is called the (complex) *rank* of $\mathfrak{g}$. There are only finitely many conjugate classes of Cartan subalgebras [22, p.395]. Each Cartan subalgebra in $\mathfrak{g}$ is G-conjugate [22, Theorem 2] to another Cartan subalgebra $\mathfrak{c} = \mathfrak{c}_\mathfrak{k} \dot{+} \mathfrak{c}_\mathfrak{p}$, where $\mathfrak{c}_\mathfrak{k} := \mathfrak{c} \cap \mathfrak{k}$ is called the toral part, and $\mathfrak{c}_\mathfrak{p} := \mathfrak{c} \cap \mathfrak{p}$ is called the vector part, such that $\mathfrak{a}_\mathfrak{k} \subset \mathfrak{c}_\mathfrak{k}$ and $\mathfrak{c}_\mathfrak{p} \subset \mathfrak{a}_\mathfrak{p}$. Such $\mathfrak{c}$ is called a *standard* Cartan subalgebra (relative to θ and $\mathfrak{a}_\mathfrak{p}$) [21, p.405].

A result of Mostow [16, Theorem 4.1] asserts that each maximal solvable subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ contains a Cartan subalgebra $\mathfrak{c}$, for example, compact Cartan subalgebra of $\mathfrak{g}$ is maximal solvable [16, Lemma 4.1]. If $\mathfrak{c}$ is a standard Cartan subalgebra, such $\mathfrak{s}$ is called a *standard* maximal solvable subalgebra (with respect to $\mathfrak{a}_\mathfrak{p}$ and θ). Each G-conjugate of $\mathfrak{s}$ is still a maximal solvable subalgebra, due to Cartan's criterion of solvablity [12, Proposition 1.43], and the adjoint action of G respects the bracket and preserves the Killing form. Thus each conjugacy class of maximal solvable subalgebras under the adjoint action of G contains a standard maximal solvable subalgebra $\mathfrak{s}$.

For example, when $\mathfrak{g} = \mathfrak{sl}(3,\mathbb{R})$, there are two conjugacy classes of Cartan subalgebras [12] represented by the standard Cartan subalgebras:

$$\mathfrak{c}_1 = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -a-b \end{pmatrix} : a, b \in \mathbb{R} \right\}, \qquad \mathfrak{c}_2 = \left\{ \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & -2a \end{pmatrix} : a, b \in \mathbb{R} \right\}.$$

However there are three conjugacy classes of maximal solvable subalgebras [18] represented by the standard maximal solvable subalgebras:

$$\begin{aligned} \mathfrak{s}_1 &= \left\{ \begin{pmatrix} a & c & e \\ 0 & b & d \\ 0 & 0 & -a-b \end{pmatrix} : a, b, c, d, e \in \mathbb{R} \right\}, \\ \mathfrak{s}_2 &= \left\{ \begin{pmatrix} a & b & c \\ -b & a & d \\ 0 & 0 & -2a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}, \\ \mathfrak{s}_3 &= \left\{ \begin{pmatrix} -2a & c & d \\ 0 & a & b \\ 0 & -b & a \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}. \end{aligned}$$

Notice that $\mathfrak{s}_2$ and $\mathfrak{s}_3$ contain conjugate standard Cartan subalgebras corresponding to $\mathfrak{c}_2$.

With the above terminology, we state Theorem 5.2 as

Theorem 5.5. *Let S be the set of standard maximal solvable subalgebra with respect to $\mathfrak{a}_\mathfrak{p}$ and θ. Let $\beta \in \mathfrak{a}_\mathfrak{p}$. Then*

$$\pi(\mathfrak{k} + K \cdot \beta) \cap \mathfrak{b}) = \pi((\mathfrak{k} + K \cdot \beta) \cap (\cup_{\mathfrak{s} \in S} \mathfrak{s})) = conv\, W\beta.$$

Remark 5.6. In generally it is *not* true that given arbitrary $x \in \mathfrak{g}$, $K \cdot x$ intersects $\mathfrak{s}$ for some standard maximal solvable subalgebra $\mathfrak{s}$ of the real semisimple algebra $\mathfrak{g}$. Consider the real simple Lie algebra $\mathfrak{g} = \mathfrak{su}_{1,1}$.

Proof. We consider the group:

$$SU(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \overline{\beta} & \overline{\alpha} \end{pmatrix} : |\alpha|^2 - |\beta|^2 = 1 \right\},$$

whose Lie algebra is a real form of $\mathfrak{sl}(2,\mathbb{C})$:

$$\begin{aligned} \mathfrak{su}_{1,1} &= \left\{ \begin{pmatrix} ia & c \\ \overline{c} & -ia \end{pmatrix} : a \in \mathbb{R}, c \in \mathbb{C} \right\}, \\ K &= \left\{ \operatorname{diag}(e^{i\theta}, e^{-i\theta}) : \theta \in \mathbb{R} \right\}, \\ \mathfrak{k} &= \left\{ \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix} : a \in \mathbb{R} \right\}, \\ \mathfrak{p} &= \left\{ \begin{pmatrix} 0 & c \\ \overline{c} & 0 \end{pmatrix} : c \in \mathbb{C} \right\}, \\ \mathfrak{a}_\mathfrak{p} &= \left\{ \begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} : b \in \mathbb{R} \right\}. \end{aligned}$$

There are two conjugate standard Cartan subalgebras: $\mathfrak{k}$ and $\mathfrak{a}_\mathfrak{p}$ [22, p.401]. They are also the two [19, p.518] standard maximal solvable subalgebras of $\mathfrak{su}_{1,1}$. Since $\mathfrak{k}$ is a compact Cartan subalgebra of $\mathfrak{su}_{1,1}$, it is maximal solvable [17, Lemma 4.1]. To see $\mathfrak{a}_\mathfrak{p}$ is maximal solvable, let $\mathfrak{s}$ be a standard maximal solvable subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}_{\mathfrak{p}}$. Now $\mathfrak{s}$ cannot contain (1-dimensional) $\mathfrak{k}$ since $\mathfrak{k}$ is a standard maximal solvable subalgebra. So $\mathfrak{s}$ is either $\mathfrak{a}_{\mathfrak{p}}$ itself or $\mathfrak{p}$ but the later is not an algebra. So $\mathfrak{s} = \mathfrak{a}_{\mathfrak{p}}$. Clearly no element in K sends

$$\begin{pmatrix} ia & c \\ \overline{c} & -ia \end{pmatrix} \in \mathfrak{su}_{1,1}, \quad a, c \neq 0,$$

into either $\mathfrak{k}$ or $\mathfrak{a}_{\mathfrak{p}}$. $\square$

References

[1] A.R. Amir-Moéz and A. Horn, Singular values of a matrix, Amer. Math. Monthly, **65** (1958) 742–748.

[2] R. Bhatia, *Matrix Analysis*, Springer, New York, 1997.

[3] I. Bendixson, Sur les racines d'une équation fondementale, Acta Math. **5** (1902), 359–365.

[4] J.T.I'A. Bromwich, On the roots of characteristic equation of a linear substitution, Acta Math., **30** (1906), 295–304.

[5] D.Z. Djoković and T.Y. Tam, Some questions about semisimple Lie groups originating in matrix theory, to appear in Canadian Mathematical Bulletin.

[6] K. Fan, Maximal properties and inequalities for the eigenvalues of completely continuous operators, Proc. Nat. Acad. Sci. U.S.A. **37** (1951), 760–766.

[7] W. Fulton and p. Pragacz, *Schubert Varieties and Degeneracy Loci*, Lecture Notes in Mathematics 1689, Springer, 1998.

[8] S. Helgason, Differential Geometry, Lie Groups, and Symmetric Spaces, Academic Press, New York, 1978.

[9] A. Hirsch, Sur les racines d'une équation fondamentale, Acta math. **25** (1905), 367–370.

[10] R.A. Horn and C.R. Johnson, *Topics in Matrix Analysis*, Cambridge Univ. Press, 1991.

[11] J.E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Springer-Verlag, New York 1972.

[12] A.W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[13] B. Kostant, On the conjugacy of real Cartan subalgebras. I. Proc. Nat. Acad. Sci. U. S. A. **41** (1955), 967–970.

[14] B. Kostant, On convexity, the Weyl group and Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup. (4), **6** (1973), 413–460.

[15] A.W. Marshall and I. Olkin *Inequalities: Theory of Majorization and Its Applications*, Academic Press, New York, 1979.

[16] G.D. Mostow, On maximal subgroups of real Lie groups, Ann. of Math. (2) **74** (1961), 503–517.

[17] L. Mirsky, Matrices with prescribed characteristic roots and diagonal elements, J. London Math. Soc., **33** (1958), 14–21.

[18] M. Perroud, The maximal solvable subalgebras of the real classical Lie algebras, J. Mathematical Phys. **17** (1976), 1028–1033.

[19] M. Perroud, The maximal solvable sugalgebras of the real classical Lie algebras, II, Group theoretical methods in physics (Fourth Internat. Colloq., Nijmegen, 1975), Lecture Notes in Phys., Vol. 50, Springer, Berlin, (1976), 516–522.

[20] S. Sherman and C.J. Thompson, Equivalences on eigenvalues, Indiana Univ. Math. J., **21** (1972), 807–814.

[21] L.P. Rothschild, Orbits in a real reductive Lie algebra, Trans. Amer. Math. Soc., **168** (1972), 403–421.

[22] M. Sugiura, Conjugate classes of Cartan subalgebras in real semisimple Lie algebras, J. Math. Soc. Japan, **11**(1959), 374–434.

[23] T.Y. Tam, A unified extension of two result of Ky Fan on the sum of matrices, Proc. Amer. Math. Soc., **126** (1998), 2607–2614.

[24] T.Y. Tam, A Lie theoretical approach of Thompson's theorems of singular values-diagonal elements and some related results, J. of London Math. Soc. (2), **60** (2002), 431–448.

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