Unspecified Journal
Volume 00, Number 0, Pages 000–000
S ????-????(XX)0000-0

ALUTHGE ITERATION IN SEMISIMPLE LIE GROUP

HUAJUN HUANG AND TIN-YAU TAM

Abstract. We extend, in the context of connected noncompact semisimple Lie group, two results of Antezana, Massey, and Stojanoff: Given $0 < \lambda < 1$, (a) the limit points of the sequence $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ are normal, and (b) $\lim\limits_{m\to\infty} \|\Delta_\lambda^m(X)\| = r(X)$, where $\|X\|$ is the spectral norm and $r(X)$ is the spectral radius of $X \in \mathbb{C}_{n\times n}$ and $\Delta_\lambda(X)$ is the λ-Aluthge transform of X.

1. Introduction

Given $0 < \lambda < 1$, the λ-Aluthge transform of $X \in \mathbb{C}_{n\times n}$ [4]:

$$\Delta_\lambda(X) := P^\lambda U P^{1-\lambda}$$

has been extensively studied, where $X = UP$ is the polar decomposition of X, that is, U is unitary and P is positive semidefinite. The spectrum (counting multiplicities) remains fixed under Δ_λ:

$$\sigma \overset{m}{=} \sigma(\Delta_\lambda(X)), \tag{1.1}$$

since $X = (UP^{1-\lambda})P^\lambda$ and $\Delta_\lambda(X) = P^\lambda(UP^{1-\lambda})$, and Δ_λ respects unitary similarity:

$$\Delta_\lambda(VXV^{-1}) = V\Delta_\lambda(X)V^{-1}, \quad V \in U(n). \tag{1.2}$$

Let $\|\cdot\|$ denote the spectral norm and $r(\cdot)$ denote the spectral radius. Define $\Delta_\lambda^m(X) := \Delta_\lambda(\Delta_\lambda^{m-1}(X))$ inductively, $m \in \mathbb{N}$. Notice that $\{\|\Delta_\lambda^m(X)\|\}_{m\in\mathbb{N}}$ is monotonic decreasing since

$$\|\Delta_\lambda(X)\| = \|P^\lambda U P^{1-\lambda}\| \le \|P^\lambda\|\|U\|\|P^{1-\lambda}\| = \|P\| = \|X\|$$

so that $\lim_{m\to\infty} \|\Delta_\lambda^m(X)\|$ exists. Because (1.1)

$$r(X) \le \|\Delta_\lambda(X)\| \le \|X\|. \tag{1.3}$$

2000 Mathematics Subject Classification. Primary 22E46, 15A45

Key words. Aluthge iteration, complete multiplicative Jordan decomposition, semisimple Lie group

It is known that the sequence of matrices $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ converges if (a) the eigenvalues of X have distinct moduli [9] or (b) if X is diagonalizable and $\lambda = 1/2$ [2]. Very recently Antezana, Pujals and Stojanoff [1] proved that $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ converges if X is diagonalizable but the general case is still open. The convergence problem for $\mathbb{C}_{n\times n}$ is reduced to $\mathrm{GL}_n(\mathbb{C})$ [6] and is further reduced to $\mathrm{SL}_n(\mathbb{C})$ since

$$\Delta_\lambda(cX) = c\Delta_\lambda(X), \qquad c \in \mathbb{C}.$$

Recently [6] Antezana, Massey and Stojanoff proved the following two results.

Theorem 1.1. [6] Let $X \in \mathbb{C}_{n\times n}$ and $0 < \lambda < 1$.

(a) Any limit point of the λ-Aluthge sequence $\{\Delta_\lambda^m(X)\}_{m\in\mathbb{N}}$ is normal, with eigenvalues $\lambda_1(X), \ldots, \lambda_n(X)$.

(b) $\lim_{m\to\infty} \|\Delta_\lambda^m(X)\| = r(X)$, where $r(X)$ is the spectral radius of X.

When $\lambda = 1/2$, (b) is the finite dimensional case of Yamazaki's result [17] (see [15, 13] for a simpler proof of Yamazaki's result for general Hilbert space operators).

By using compound matrix as in [5] (also see [16]), one deduces from Theorem 1.1(b) that

$$\lim_{m\to\infty} s_i(\Delta_\lambda^m(X)) = |\lambda_i(X)|, \quad i = 1, \ldots, n. \tag{1.4}$$

Our goal is to extend Theorem 1.1 and to establish other related results in the context of connected noncompact semisimple Lie group.

2. Extension for semisimple Lie groups

Since our main interest is connected noncompact semisimple Lie group, we start to take a close look of (1.4) for $X \in \mathrm{GL}_n(\mathbb{C})$ from which Theorem 1.1(b) is a particular case. The equation may be interpreted as a relation between the singular value decomposition and the complete multiplicative Jordan decomposition of $X \in \mathrm{GL}_n(\mathbb{C})$. Let $A_+ \subset \mathrm{GL}_n(\mathbb{C})$ denote the set of all positive diagonal matrices with diagonal entries in nonincreasing order. Recall that the singular value decomposition of $X \in \mathrm{GL}_n(\mathbb{C})$ asserts [12, p.129] that there exist unitary matrices U, V such that

$$X = Ua_+(X)V, \tag{2.1}$$

where $a_+(X) = \mathrm{diag}\,(s_1(X), \ldots, s_n(X)) \in A_+$. Though U and V in the decomposition (2.1) are not unique, $a_+(X) \in A_+$ is uniquely defined. The complete multiplicative Jordan decomposition [7, p.430-431] of $X \in \mathrm{GL}_n(\mathbb{C})$ asserts that $X = ehu$, where e is diagonalizable with eigenvalues of modulus 1, h is diagonalizable over $\mathbb{R}$ with positive eigenvalues and $u = \exp n$, where n is nilpotent [7]. The eigenvalues of h are the moduli of the eigenvalues of X, counting multiplicities. The elements e, h, u commute with each others and are uniquely defined. Moreover h is conjugate to a unique element in A_+, namely, $b(X) = \operatorname{diag}(|\lambda_1(X)|, \ldots, |\lambda_n(X)|) \in A_+$. Thus (1.4) may be rewritten as

$$\lim_{m\to\infty} a_+(\Delta_\lambda^m(X)) = b(X), \quad X \in \mathrm{GL}_n(\mathbb{C}),$$

or simply

$$\lim_{m\to\infty} a_+(\Delta_\lambda^m(X)) = b(X), \quad X \in \mathrm{SL}_n(\mathbb{C}). \tag{2.2}$$

since $\Delta_\lambda(cX) = c\Delta_\lambda(X)$, $a_+(cX) = |c|a_+(X)$, $b(cX) = |c|b(X)$.

The Lie group extension involves a generalized Aluthge iteration, and two decompositions alike singular value decomposition and Jordan decomposition, namely, Cartan decomposition and complete multiplicative Jordan decomposition.

Let G be a connected noncompact semisimple Lie group having $\mathfrak{g}$ as its Lie algebra. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a fixed Cartan decomposition of $\mathfrak{g}$. Let $K \subset G$ be the subgroup with Lie algebra $\mathfrak{k}$. Set

$$P := \exp \mathfrak{p}.$$

The well known Cartan decomposition [7] asserts that the map $K \times P \to G$, $(k, p) \mapsto kp$ is a diffeomorphism. In particular

$$G = KP$$

and every element $g \in G$ can be uniquely written as

$$g = kp, \qquad k \in K,\ p \in P. \tag{2.3}$$

Given $0 < \lambda < 1$, the λ-Aluthge transform of $\Delta_\lambda : G \to G$ is defined as

$$\Delta_\lambda(g) := p^\lambda k p^{1-\lambda},$$

where $p^\lambda := e^{\lambda x} \in P$ if $p = e^x$ for some $x \in \mathfrak{p}$. The map $\Theta : G \to G$

$$\Theta(kp) = kp^{-1}, \quad k \in K,\ p \in P,$$

is an automorphism of G [10, p.387]. The map $* : G \to G$ defined by

$$g^* := \Theta(g^{-1}) = pk^{-1}, \qquad g \in G$$

is clearly a diffeomorphism and $g^*g = p^2 \in P$. Now

$$\Delta_\lambda(g) = (g^*g)^{\lambda/2} g (g^*g)^{-\lambda/2}.$$

So we have immediately

Proposition 2.1. Let $g \in G$.

(1) $\Delta_\lambda(g)$ and g are conjugate: $\Delta_\lambda(g) = (g^*g)^{\lambda/2} g (g^*g)^{-\lambda/2}$.
(2) $\Delta_\lambda(vgv^{-1}) = v\Delta_\lambda(g)v^{-1}$, for all $v \in K$.
(3) The map $(0,1)\times G \to G$ such that $(\lambda, g) \mapsto \Delta_\lambda(g)$ is C^∞. Thus $\Delta_\lambda : G \to G$ is C^∞.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace and set $A := \exp \mathfrak{a}$. Fix a *closed* Weyl chamber $\mathfrak{a}_+$ in $\mathfrak{a}$ and set $A_+ := \exp \mathfrak{a}_+$. By using the fact [10, p.320] that

$$\mathfrak{p} = \cup_{k\in K} \mathrm{Ad}\,(k)\mathfrak{a}, \tag{2.4}$$

each element $p \in P$ is conjugate via K to a unique element in A_+, denoted by $a_+(p)$. So (2.3) can be written as $g = kua_+(p)u^{-1}$, $u \in K$ and thus [7, p.402]

$$G = KA_+K. \tag{2.5}$$

Though $k_1, k_2 \in K$ in $g = k_1 a_+(g) k_2$ are not unique, the element $a_+(g) \in A_+$ is unique.

An element $h \in G$ is called *hyperbolic* if $h = \exp X$ where $X \in \mathfrak{g}$ is real semisimple, that is, $\mathrm{ad}\, X \in \mathrm{End}\, \mathfrak{g}$ is diagonalizable over $\mathbb{R}$. An element $u \in G$ is called *unipotent* if $u = \exp N$ where $N \in \mathfrak{g}$ is nilpotent, that is, $\mathrm{ad}\, N \in \mathrm{End}\, \mathfrak{g}$ is nilpotent. An element $e \in G$ is *elliptic* if $\mathrm{Ad}\, e \in \mathrm{Aut}\, \mathfrak{g}$ is diagonalizable over $\mathbb{C}$ with eigenvalues of modulus 1. The complete multiplicative Jordan decomposition (CMJD) [11, Proposition 2.1] for G asserts that each $g \in G$ can be uniquely written as

$$g = ehu, \tag{2.6}$$

where e is elliptic, h is hyperbolic and u is unipotent and the three elements e, h, u commute. We write $g = e(g)h(g)u(g)$.

It turns out that $h \in G$ is hyperbolic if and only if it is conjugate to a unique element $b(h) \in A_+$ [11, Proposition 2.4]. Denote

$$b(g) := b(h(g)). \tag{2.7}$$

Clearly $b(g) = a_+(g)$ if $g \in A$ and $b(g) = a_+(g) = g$ if $g \in A_+$.

When $G = \mathrm{SL}_n(\mathbb{C})$, $b(X) = \mathrm{diag}\,(|\lambda_1(X)|, \ldots, |\lambda_n(X)|)$ if A_+ is chosen to be the set of diagonal matrices in $\mathrm{SL}_n(\mathbb{C})$ of descending diagonal entries. Similar to (1.1) we have the following result.

Proposition 2.2. Suppose that G is a connected noncompact semisimple Lie group. For any $g \in G$, $0 < \lambda < 1$, $b(g) = b(\Delta_\lambda(g))$.

Proof. By Proposition 2.1, $\Delta_\lambda(g)$ and g are conjugate, say, $\Delta_\lambda(g) = ygy^{-1}$. Let $g = ehu$ be the CMJD of g. Then

$$ygy^{-1} = (yey^{-1})(yhy^{-1})(yuy^{-1})$$

is the CMJD of ygy^{-1} so that $h(ygy^{-1}) = yh(g)y^{-1}$, that is, the hyperbolic parts of g and $\Delta_\lambda(g)$ are conjugate. Thus the desired result follows from [11, Proposition 2.4]. ☐

Denote by $I(G)$ the set of irreducible representations of G, V_π the representation space of $\pi \in I(G)$. For each $\pi \in I(G)$, there is an inner product structure [11, p.435] such that

(1) $\pi(k)$ is unitary for all $k \in K$,
(2) $\pi(p)$ is positive definite for all $p \in P$.

We will assume that V_π is endowed with this inner product. We have the matrix λ-Aluthge transform $\Delta_\lambda : \mathrm{GL}(V_\pi) \to \mathrm{GL}(V_\pi)$ (abuse of notation) as well as $\Delta_\lambda : G \to G$.

Lemma 2.3. Suppose that G is a connected noncompact semisimple Lie group. For any $\pi \in I(G)$ and $0 < \lambda < 1$,

$$\pi \circ \Delta_\lambda = \Delta_\lambda \circ \pi,$$

where Δ_λ on the left is the Aluthge transform of $g \in G$ with respect to the Cartan decomposition $G = KP$ and that on the right is the matrix Aluthge transform of $\pi(g) \in \mathrm{GL}(V_\pi)$ with respect to the polar decomposition.

Proof. Let $g = kp$, $k \in K, p \in P$. Then

$$\begin{aligned} \Delta_\lambda \circ \pi(g) &= \Delta_\lambda \circ \pi(kp) = \Delta_\lambda(\pi(k)\pi(p)) = \pi(p)^\lambda \pi(k)\pi(p)^{1-\lambda} \\ &= \pi(p^\lambda)\pi(k)\pi(p^{1-\lambda}) = \pi(p^\lambda k p^{1-\lambda}) = \pi \circ \Delta_\lambda(g). \end{aligned}$$

☐

An element $g \in G$ is said to be *normal* if $kp = pk$, where $g = kp$ ($k \in K$ and $p \in P$) is the Cartan decomposition of g.

The following is an extension of the well-known result that $X \in \mathbb{C}_{n\times n}$ is normal if and only if the singular values of X and the eigenvalue moduli are identical, counting multiplicities.

Proposition 2.4. Let G be a connected noncompact semisimple Lie group. Then $g \in G$ is normal if and only if $b(g) = a_+(g)$.

Proof. Suppose that g is normal, that is, $kp = pk$. Since k is elliptic and p is hyperbolic [11, Proposition 2.3, Proposition 2.4], $g = kp \cdot 1$ is the CMJD of g, i.e., $h(g) = p$. Thus $b(p) = b(h(g)) = b(g)$. Since each element $p \in P$ is conjugate via K to a unique element in A_+, Thus $a_+(g) = a_+(p) = b(p)$. Hence $b(g) = a_+(g)$.

Conversely, suppose $a_+(g) = b(g)$. Let $g = ehu$ be the CMJD of g. By [11, Proposition 3.4] and its proof

$$\mathrm{Ad}\, g = \mathrm{Ad}\, e \,\mathrm{Ad}\, h \,\mathrm{Ad}\, u$$

is the CMJD of $\mathrm{Ad}\, g \in \mathrm{SL}(\mathfrak{g})$ so that the eigenvalue moduli of $\mathrm{Ad}\, g$ are the eigenvalues of $\mathrm{Ad}\, h$. Since $b(g)$ and h are conjugate, the eigenvalues of $\mathrm{Ad}\, b(g)$ are the eigenvalue moduli of $\mathrm{Ad}\, g$.

Let $g = k_1 a_+(g) k_2$ be the KA_+K decomposition of g. Then

$$\mathrm{Ad}\, g = \mathrm{Ad}\, k_1 \mathrm{Ad}\, a_+(g) \mathrm{Ad}\, k_2,$$

where $\mathrm{Ad}\, k_1$ and $\mathrm{Ad}\, k_2$ are unitary matrices and $\mathrm{Ad}\, a_+(g)$ is a positive definite matrix whose eigenvalues are the singular values of $\mathrm{Ad}\, g$. Clearly $b(g) = a_+(g)$ implies that $\mathrm{Ad}\, b(g) = \mathrm{Ad}\, a_+(g)$. Thus the eigenvalue moduli of the matrix $\mathrm{Ad}\, g$ and the singular values of $\mathrm{Ad}\, g$ are identical. So $\mathrm{Ad}\, g$ is a normal matrix. Let $g = kp$ be the Cartan decomposition of G, $k \in K$, $p \in P$. Now $\mathrm{Ad}\, g = \mathrm{Ad}\, k\, \mathrm{Ad}\, p$ is the Cartan (polar) decomposition of the matrix $\mathrm{Ad}\, g$. So $\mathrm{Ad}\, k\, \mathrm{Ad}\, p = \mathrm{Ad}\, p\, \mathrm{Ad}\, k$ because $\mathrm{Ad}\, g$ is a normal matrix, that is, $\mathrm{Ad}\,(kp) = \mathrm{Ad}\,(pk)$. Then $kp = pkc$ where $c \in Z \subset K$ and Z is the center of G. We have $kpk^{-1} = pc$. Since $kpk^{-1} \in P$ by (2.4). So kpk^{-1} is hyperbolic since the elements in $\mathfrak{p}$ are real semisimple. By the uniqueness of CMJD, c is the identity element and $kp = pk$. $\square$

The second statement of the following result becomes the continuity of the singular values of nonsingular matrices when $G = \mathrm{SL}_n(\mathbb{C})$ since the determinant is a continuous function.

Proposition 2.5. Suppose that G is a connected noncompact semisimple Lie group. The following maps are continuous.

(1) $a'_+ : \mathfrak{p} \to \mathfrak{a}_+$ where for each $X \in \mathfrak{p}$, $a'_+(X)$ is the unique element in $\mathfrak{a}_+$ such that $a'_+(X) = \mathrm{Ad}\,(s)X$ for some $s \in K$. Indeed it is a contraction.

(2) $a_+ : G \to A_+$ where $a_+(g) \in A_+$ is the unique element in $g = k_1 a_+(g) k_2 \in G$, where $k_1, k_2 \in K$.

Proof. (1) It is known that [8, Cor 3.10] $a'_+(X) - a'_+(Y) \in \mathrm{conv}\, W a'_+(X - Y)$ for any $X, Y \in \mathfrak{p}$. Let $\|\cdot\|$ be the norm on $\mathfrak{p}$ induced by the Killing form of $\mathfrak{g}$. Since $\|\cdot\|$ is K-invariant and strictly convex, $\|a'_+(X) - a'_+(Y)\| \le \|a'_+(X - Y)\| = \|X - Y\|$. So the map a'_+ is a contraction and thus continuous.

(2) Since the map $G \to P$ such that $g = kp \mapsto p$ is differentiable and $a_+(g) = a_+(p)$, it suffices to establish the continuity of $a_+ : P \to A_+$. The map $\exp : \mathfrak{p} \to P$ is a diffeomorphism onto and the inverse $\log : P \to \mathfrak{p}$ is well defined. Now $a_+ = \exp \circ a'_+ \circ \log$ on P is clearly continuous. $\square$

We remark that CMJD is *not* continuous. For example, if $G = \mathrm{SL}_2(\mathbb{C})$, consider $g_\epsilon = \begin{pmatrix} 1+\epsilon & 1 \\ 0 & 1/(1+\epsilon) \end{pmatrix}$ with $\epsilon \geq 0$. As $\epsilon > 0$, $e(g_\epsilon)$ and $u(g_\epsilon)$ are the identity matrix and $h(g_\epsilon) = g_\epsilon$. However, when $\epsilon = 0$, $e(g_0)$ and $h(g_0)$ are the identity matrix but $u(g_0) = g_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Though the function h is not continuous, $b : G \to A_+$ is continuous.

Theorem 2.6. If G is a connected noncompact semisimple Lie group, then $b : G \to A_+$ is continuous.

Proof. Since $\exp : \mathfrak{a} \to A$ is bijective, $\log b(g)$ is well-defined. Denote

$$A(g) := \exp(\operatorname{conv}(W \log b(g))), \tag{2.8}$$

where $\operatorname{conv} Wx$ denotes the convex hull of the orbit of $x \in \mathfrak{a}$ under the action of the Weyl group of $(\mathfrak{g}, \mathfrak{a})$, which may be defined as the quotient of the normalizer of A in K modulo the centralizer of A in K. Notice that $A(g) = A(b(g))$ is compact in G since the Weyl group W is finite.

We first claim that

$$A(g) \subset A(a_+(g)), \quad g \in G. \tag{2.9}$$

To prove the claim, write $g = k_1 a_+(g) k_2$, where $a_+(g) \in A_+$, $k_1, k_2 \in K$. Then for any $\pi \in I(G)$, since $\pi(g)$ is an operator,

$$r(\pi(g)) \leq \|\pi(g)\|.$$

On the other hand

$$\begin{aligned} \|\pi(g)\| &= \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1)\pi(a_+(g))\pi(k_2)\| \\ &= \|\pi(a_+(g))\| = r(\pi(a_+(g))), \end{aligned} \tag{2.10}$$

since the spectral norm $\|\cdot\|$ is invariant under unitary equivalence, $\|B\| = r(B)$ for every positive definite endomorphism B, $\pi(k)$ is unitary and $\pi(a_+(g))$ is positive definite. So

$$r(\pi(g)) \leq r(\pi(a_+(g))).$$

A result of Kostant [11, Theorem 3.1] asserts that if $f, g \in G$, then $A(f) \supset A(g)$ if and only if $r(\pi(f)) \geq r(\pi(g))$ for all $\pi \in I(G)$, where $A(g)$ is defined in (2.8). Thus we establish (2.9).

To prove the continuity of $b : G \to A_+$, suppose $g_n \to g_0$ in G. Then by Proposition 2.5 $a_+(g_n) \to a_+(g_0)$ so that $C := \overline{\cup_{n=1}^\infty A(a_+(g_n))}$ is compact.

By (2.9) $b(g_n) \in A(g_n) \subset A(a_+(g_n)) \subset C$ so that the sequence $\{b(g_n)\}_{n\in\mathbb{N}} \subset A_+$ is contained in the compact set $C \cap A_+$. Let $\ell \in A_+$ be any arbitrary limit point of $\{b(g_n)\}_{n\in\mathbb{N}}$, i.e., $b(g_{n_k}) \to \ell$ as $k \to \infty$. We are going to show that $\ell = b(g_0)$. To this end, consider $\pi \in I(G)$ and clearly $\pi(b(g_{n_k})) \to \pi(\ell)$. We now view each $\pi(g)$, $g \in G$, as an element in $\mathrm{SL}(V_\pi)$ which may be identified with $\mathrm{SL}_n(\mathbb{C})$, where $n := \dim V_\pi$. Write the CMJD of $\pi(g) \in \mathrm{SL}_n(\mathbb{C})$ as $\pi(g) = e(\pi(g))h(\pi(g))u(\pi(g))$.

Since spectral radius is continuous on $\mathrm{SL}_n(\mathbb{C})$ [12],

$$r(\pi(b(g_{n_k})) \to r(\pi(\ell)). \tag{2.11}$$

From [11, Proposition 3.4] and its proof,

$$h(\pi(g)) = \pi(h(g)), \quad g \in G,$$

i.e., h and π commute. Since $h(g_{n_k})$ and $b(g_{n_k})$ are conjugate and since the eigenvalues of $h(\pi(g_{n_k}))$ are the eigenvalue moduli of $\pi(g_{n_k})$,

$$r(\pi(b(g_{n_k})) = r(\pi(h(g_{n_k})) = r(h(\pi(g_{n_k})) = r(\pi(g_{n_k})). \tag{2.12}$$

So from (2.11) and (2.12)

$$r(\pi(g_{n_k})) \to r(\pi(\ell)).$$

On the other hand $g_{n_k} \to g_0$ implies that

$$r(\pi(g_{n_k})) \to r(\pi(g_0)).$$

But

$$r(\pi(g_0)) = r(h(\pi(g_0)) = r(\pi(h(g_0)) = r(\pi(b(g_0))),$$

as $h(g_0)$ and $b(g_0)$ are conjugate. So

$$r(\pi(\ell)) = r(\pi(b(g_0))), \qquad \pi \in I(G).$$

By [11, Theorem 3.1], $A(\ell) = A(b(g_0))$ so that $\ell = b(g_0)$ since $\ell, b(g_0) \in A_+$. Thus $b : G \to A_+$ is continuous. ☐

We are going to extend Theorem 1.1. The roles of spectral norm and radius are played respectively by $a_+(g)$ and $b(g)$. Given any $g \in G$, we consider the sequence $\{a_+(\Delta_\lambda^m(g))\}_{m\in\mathbb{N}}$, with $\Delta_\lambda^0(g) := g$.

Theorem 2.7. Let G be a connected noncompact semisimple Lie group and $g \in G$.

(a) If G has finite center, then the sequence $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ is contained in a compact set and the limit points of the sequence $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ are normal.
(b) $\lim_{m\to\infty} a_+(\Delta_\lambda^m(g)) = b(g)$.

Proof. (b) By (1.3) and Lemma 2.3, for any $\pi \in I(G)$, $g \in G$, $m \in \mathbb{N}$,

$$\|\pi(g)\| \geq \|\Delta_\lambda^m(\pi(g))\| = \|\pi(\Delta_\lambda^m(g))\|.$$

Write $g = k_1 a_+(g) k_2$, where $a_+(g) \in A_+$, $k_1, k_2 \in K$. Then

$$\begin{aligned} \|\pi(g)\| &= \|\pi(k_1 a_+(g) k_2)\| = \|\pi(k_1)\pi(a_+(g))\pi(k_2)\| \\ &= \|\pi(a_+(g))\| = r(\pi(a_+(g))). \end{aligned} \tag{2.13}$$

Thus

$$r(\pi(a_+(g))) \geq r(\pi(a_+[\Delta_\lambda^m(g)])).$$

By [11, Theorem 3.1]

$$A(a_+(g)) \supset A(a_+(\Delta_\lambda^m(g))) \tag{2.14}$$

so that $\{a_+(\Delta_\lambda^m(g))\}_{n\in\mathbb{N}}$ is contained in the compact set $A(a_+(g))$. Let $\ell \in A(a_+(g)) \cap A_+$ be any limit point of the sequence $\{a_+[\Delta_\lambda^m(g)]\}_{m\in\mathbb{N}} \subset A(a_+(g)) \cap A_+$, namely $\lim_{i\to\infty} a_+[\Delta_\lambda^{m_i}(g)] = \ell$. Since r and π are continuous,

$$\lim_{i\to\infty} r(\pi(a_+[\Delta_\lambda^{m_i}(g)])) = r(\pi(\ell)).$$

But from (2.13),

$$r(\pi(a_+[\Delta_\lambda^{m_i}(g)])) = \|\pi(\Delta_\lambda^{m_i}(g))\| = \|\Delta_\lambda^{m_i}(\pi(g))\|$$

so that from Theorem 1.1(b) we have

$$\lim_{i\to\infty} r(\pi(a_+[\Delta_\lambda^{m_i}(g)])) = r(\pi(g)).$$

Thus $r(\pi(\ell)) = r(\pi(g))$ for all $\pi \in I(G)$, which implies that $A(\ell) = A(g) = A(b(g))$ by the result of Kostant [11, Theorem 3.1] again. Both ℓ and $b(g)$ are in A_+. Thus $\ell = b(g)$ and

$$\lim_{m\to\infty} a_+[\Delta_\lambda^m(g)] = b(g).$$

(a) In the proof of (b) we showed that $\{a_+(\Delta_\lambda^m(g))\}_{n\in\mathbb{N}}$ is contained in the compact set $A(a_+(g))$. Now each $\Delta_\lambda^m(g)$ is contained in $KA(a_+(g))K$ by (2.5).

If G has finite center, then K is compact [10, p.305] so that the sequence $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$ is contained in the compact set $KA(a_+(g))K$.

Suppose that ℓ is a limit point of $\{\Delta_\lambda^m(g)\}_{m\in\mathbb{N}}$, that is, $\lim_{i\to\infty} \Delta_\lambda^{m_i}(g) = \ell$. On one hand,

$$\lim_{i\to\infty} a_+(\Delta_\lambda^{m_i}(g)) = a_+(\lim_{i\to\infty} \Delta_\lambda^{m_i}(g)) = a_+(\ell)$$

by Proposition 2.5(2). On the other hand,

$$b(g) = \lim_{i\to\infty} b(\Delta_\lambda^{m_i}(g)) = b(\lim_{i\to\infty} \Delta_\lambda^{m_i}(g)) = b(\ell)$$

by Theorem 2.6 and Proposition 2.2. From (b) $\lim_{m\to\infty}[a_+(\Delta_\lambda^m(g))] = b(g)$ so that $a_+(\ell) = b(\ell)$. Thus by Proposition 2.4 we have the desired result.

□

References

[1] J. Antezana, E. Pujals and D. Stojanoff, Convergence of the iterated Aluthge transform sequence for diagonalizable matrices. II. λ-Aluthge transform, Integral Equations Operator Theory **62** (2008), 465–488.

[2] J. Antezana, E. Pujals and D. Stojanoff, Convergence of iterated Aluthge transform sequence for diagonalizable matrices, Adv. Math. **216** (2007), 255–278.

[3] A. Aluthge, On p-hyponormal operators for $0 < p < 1$, Integral Equations Operator Theory, **13** (1990), 307–315.

[4] A. Aluthge, Some generalized theorems on p-hyponormal operators, Integral Equations Operator Theory **24** (1996), 497–501.

[5] T. Ando, Aluthge transforms and the convex hull of the eigenvalues of a matrix, Linear and Multilinear Algebra, **52** (2004), 281–292.

[6] J. Antezana, P. Massey and D. Stojanoff, λ-Aluthge transforms and Schatten ideas, Linear Algebra Appl., **405** (2005), 177–199.

[7] S. Helgason, *Differential Geometry, Lie Groups, and Symmetric Spaces*, Academic Press, New York, 1978.

[8] R.R. Holmes and T.Y. Tam, Distance to the convex hull of an orbit under the action of a compact Lie group, J. Austral. Math. Soc. Ser. A, **66** (1999), 331–357.

[9] H. Huang and T.Y. Tam, On the convergence of Aluthge sequence, Operators and Matrices, **1** (2007), 121–142.

[10] A.W. Knapp, *Lie Groups Beyond an Introduction*, Birkhäuser, Boston, 1996.

[11] B. Kostant, On convexity, the Weyl group and Iwasawa decomposition, Ann. Sci. Ecole Norm. Sup. (4), **6** (1973), 413–460.

[12] D. Serre, *Matrices: Theory and Applications*, Springer, New York, 2002.

[13] T.Y. Tam, λ-Aluthge iteration and spectral radius, Integral Equations Operator Theory **60** (2008), 591–596.

[14] T.Y. Tam and H. Huang, An extension of Yamamoto's theorem on the eigenvalues and singular values of a matrix, J. Math. Soc. Japan, **58** (2006), 1197–1202.

[15] D. Wang, Heinz and McIntosh inequalities, Aluthge transformation and the spectral radius, Math. Inequal. Appl. **6** (2003), 121–124.

[16] T. Yamamoto, Extreme values of the roots of matrices, J. Math. Soc. Japan, **19** (1967), 171–178.

[17] T. Yamazaki, An expression of spectral radius via Aluthge transformation, Proc. Amer. Math. Soc., **130** (2002), 1131–1137.

Department of Mathematics and Statistics, Auburn University, AL 36849–5310, USA
E-mail address: `huanghu@auburn.edu, tamtiny@auburn.edu`