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ADDITIVE COMPOUND MATRICES AND REPRESENTATION OF $\mathfrak{gl}_n(\mathbb{C})$

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ABSTRACT. Schur-Horn convexity result and Cheung-Tsing star-shapedness result are extended for additive compound matrices. Generalizations in the context of symmetry classes of tensors (the irreducible characters are not necessarily linear) are discussed. Further generalizations are obtained in the context of reductive Lie group.

1. INTRODUCTION

Given an $n \times n$ complex matrix A, $1 \le k \le n$, the kth compound of A is defined as the $\binom{n}{k} \times \binom{n}{k}$ complex matrix $C_k(A)$ whose elements are defined as

$$C_k(A)_{\alpha,\beta} = \det A[\alpha|\beta] \tag{1.1}$$

where $\alpha, \beta \in Q_{k,n}$ and $Q_{k,n} = \{\alpha = (\alpha(1), \ldots, \alpha(k)) : 1 \le \alpha(1) < \cdots < \alpha(k) \le n\}$. The compound matrices are well studied [12, 13, 14] and applied to some matrix inequalities, for example, Weyl's inequalities on the eigenvalues and singular values of an $n \times n$ matrix A [1, Theorem II.3.6], [15, p.232].

The less well-known is the kth additive compound of A which is usually defined as

$$\Delta_k(A) = \frac{d}{dt}\Big|_{t=0} C_k(I + tA), \tag{1.2}$$

or equivalently $C_k(I + tA) = I + t\Delta_k(A) + t^2 R + \cdots$. It has a nice connection [18, p.858] with the differential system:

$$\frac{dx}{dt} = A(t)x, \tag{1.3}$$

where $A(t) \in C_{n\times n}$ is a continuous matrix valued function of t. If $x_1, \ldots, x_k \in \mathbb{C}_{n\times n}$ are solutions to the differential system, then $x_1 \wedge \cdots \wedge x_k$ is a solution to the additive compound system:

$$\frac{dy}{dt} = \Delta_k(A(t))y. \tag{1.4}$$

Thus for any $A \in \mathbb{C}_{n\times n}$,

$$C_k(\exp(A)) = \exp(\Delta_k(A)) \tag{1.5}$$

by considering the solutions of the systems (1.3) and (1.4) [18].

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The additive compound is also related to the k-th numerical range of A, which is introduced by Halmos:

$$W_k(A) = \{\sum_{i=1}^{k} (Ax_i, x_i) : x_1, \dots, x_k \text{ are orthonormal in } \mathbb{C}^n\},$$

in which $\mathbb{C}^n$ is endowed with an inner product. It was conjectured and proved affirmatively by Berger that $W_k(A)$ is a compact convex set in $\mathbb{C}$ [5]. It can be rewritten in the following form

$$W_k(A) = \{(\Delta_k(A)x^\wedge, x^\wedge) : x^\wedge = x_1 \wedge \cdots \wedge x_k, x_1, \dots, x_k \text{ are orthonormal in } \mathbb{C}^n\},$$

where the inner product on $\wedge^k \mathbb{C}^n$ is induced by the given inner product of $\mathbb{C}^n$ (see (2.3)). A proof can be obtained via (2.2) and the induced inner product. We will review some basic properties of the additive compound in Section 2. In Section 3 we discuss Schur-Horn's extension and Cheung-Tsing's extension. The key idea is that the two operators Δ_k and diag (taking the diagonal part of a matrix) commute. The derivative of the induced operator which is a generalized notion of additive compound is discussed in Section 4. Finally we further generalize the results in the context of Lie group in the last section.

2. Basic properties of additive compound

Equation (1.5) has a favor of Lie theory which evolved from the study of differential equations. Indeed it is pointed out in [12] that

$$\Delta_k(A) = \frac{d}{dt}\Big|_{t=0} C_k(e^{tA}). \tag{2.1}$$

We may consider C_k as a map from the general linear group $GL_n(\mathbb{C})$ (the group of $n \times n$ nonsingular complex matrices or nonsingular linear maps on $\mathbb{C}^n$ onto itself) to the Lie group $\mathrm{Aut}\,(\wedge^k C^n)$ in which the group operation is the usual composition. Now the Lie algebra of $GL_n(\mathbb{C})$ is $\mathfrak{gl}_n(\mathbb{C})$ which may be viewed as $\mathbb{C}_{n\times n}$ and the Lie algebra of $\mathrm{Aut}\,(\wedge^k C^n)$ is $\mathrm{End}\,(\wedge^k C^n)$. Now Δ_k is a representation of the Lie algebra $\mathbb{C}_{n\times n}$ with the bracket operation $[A, B] = AB - BA$ by the following result.

Proposition 2.1. *The map $C_k : GL_n(\mathbb{C}) \to \mathrm{Aut}\,(\wedge^k C^n)$ is a representation and its differential at the identity I_n is $\Delta_k : \mathbb{C}_{n\times n} \to \mathrm{End}\,(\wedge^k C^n)$. Hence Δ_k is a Lie algebra homomorphism, i.e., $\Delta_k(AB - BA) = \Delta_k(A)\Delta_k(B) - \Delta_k(B)\Delta_k(A)$, and $C_k(\exp(A)) = \exp(\Delta_k(A))$.*

Proof. Since $C_k(AB) = C_k(A)C_k(B)$ [13] and C_k is continuous and thus analytic [6, Theorem 2.6, p.117], C_k is a representation of $GL_n(\mathbb{C})$. The differential dC_k at the identity is given by [22, p.107]

$$dC_k(A) = \frac{d}{dt}\Big|_{t=0} C_k(e^{tA}),$$

which is equal to $\Delta_k(A)$ by (2.1). Then by [6, Lemma 1.12, p.110], we have the desired results. □

Remark 2.2. It is known that C_k as a representation of $GL_n(\mathbb{C})$ is reducible [16]. In general it is not faithful, for example, C_k sends I_n and

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \oplus I_{n-2}$$

to the same image $I_{\binom{n}{k}}$, $k \neq 1$. The same example shows that the representation Δ_k of $\mathfrak{gl}_n(\mathbb{C})$ is not faithful.

Viewing $\Delta_k(A)$ as a linear map on $\wedge^k\mathbb{C}^n$, it acts in the following way [12]:

$$\Delta_k(A)x_1 \wedge \cdots \wedge x_k = \sum_{i=1}^{k} x_1 \wedge \cdots \wedge Ax_i \wedge \cdots \wedge x_k. \tag{2.2}$$

Let $(\cdot,\cdot)$ be an inner product on $\mathbb{C}^n$. Then

$$(u^\wedge, v^\wedge) := \frac{1}{k!}\sum_{\sigma\in S_k}\epsilon(\sigma)\prod_{i=1}^{k}(u_i, v_{\sigma(i)}) \tag{2.3}$$

defines an (induced) inner product on $\wedge^k\mathbb{C}^n$, where $u^\wedge := u_1 \wedge \cdots \wedge u_k$. With respect to the induced inner product, if $E = \{e_1, \ldots, e_n\}$ is an orthonormal basis of $\mathbb{C}^n$, then

$$(e_\alpha^\wedge, e_\beta^\wedge) = \frac{1}{k!}\sum_{\sigma\in S_k}\epsilon(\sigma)\delta_{\alpha,\beta\sigma}, \qquad \alpha,\beta \in Q_{k,n},$$

where $e_\alpha^\wedge := e_{\alpha(1)}\wedge\cdots\wedge e_{\alpha(k)}$ and thus $E'_\wedge := \{\sqrt{k!}\, e_\alpha : \alpha \in Q_{k,n}\}$ is an orthonormal basis of $\wedge^k\mathbb{C}^n$.

If A is the matrix representation of a linear map $T : \mathbb{C}^n \to \mathbb{C}^n$ with respect to the basis (not necessarily orthonormal) E of $\mathbb{C}^n$, then the matrix representation of $\Delta_k(T)$ with respect to $E'_\wedge$ is given by Fiedler [4, Theorem 2.4]: if $A = (a_{ij}) = [T]_E^E$, then for $\alpha, \beta \in Q_{k,n}$, $([\Delta_k(T)]_{E'^\wedge}^{E'^\wedge})_{\alpha,\beta} = \Delta_k(A)$ and

$$\Delta_k(A)_{\alpha,\beta} = \begin{cases} \sum_{i=1}^k a_{\alpha(i),\alpha(i)} & \text{if } \alpha = \beta, \\ (-1)^{r+s} a_{\alpha(s),\beta(r)} & \text{if exactly one entry } \alpha(s) \text{ in } \alpha \text{ does not occur} \\ & \text{ in } \beta \text{ and } \beta(r) \text{ does not occur in } \alpha, \\ 0 & \text{otherwise, i.e., } \alpha \text{ differ from } \beta \text{ in two} \\ & \text{ or more entries.} \end{cases}$$

Certainly the matrix representation for $\Delta_k(T)$ with respect to the basis $E_\wedge := \{e_\alpha^\wedge : \alpha \in Q_{k,n}\}$ remains the same.

We list some basic properties of additive compound matrices. The third one seems to be new and follows from Proposition 2.1 immediately or (2.2). The others can be found in [12], [15, p.505-506], and [11, p.40-41]. We will extend all the properties in the last section.

Proposition 2.3. *Let $A, B \in \mathbb{C}_{n\times n}$, $1 \le k \le n$. Then*

1. $\Delta_k : \mathbb{C}_{n\times n} \to \wedge^k\mathbb{C}^n$ *is linear.*
2. $\Delta_k(SAS^{-1}) = C_k(S)\Delta_k(A)C_k(S)^{-1}$, *for any nonsingular S.*
3. $\Delta_k(AB-BA) = \Delta_k(A)\Delta_k(B)-\Delta_k(B)\Delta_k(A)$. *Thus if A and B commute, so do $\Delta_k(A)$ and $\Delta_k(B)$.*
4. $\Delta_k(A^*) = \Delta_k(A)^*$, *where the adjoint on the right side is with respect to the induced inner product.*
5. *If A is Hermitian (normal respectively), so is $\Delta_k(A)$.*
6. *The eigenvalues of $\Delta_k(A)$ are $\sum_{i=1}^k \lambda_{\alpha(i)}$, $\alpha \in Q_{k,n}$. Thus $\operatorname{tr}\Delta_k(A) = \binom{n-1}{k-1}\operatorname{tr} A$.*

By the last two parts of the Proposition 2.3, if A is psd (pd), so is $\Delta_k(A)$. Moreover $\Delta_k(I_n) = kI_{\binom{n}{k}}$.

Remark 2.4. We note that $\Delta_k(AB) \neq \Delta_k(A)\Delta_k(B)$ for general $A, B \in \mathbb{C}_{n\times n}$. The singular values of $\Delta_k(A)$ in general are not $\sum_{i=1}^k s_{\alpha(i)}(A)$, where $s_1(A) \geq \cdots \geq s_n(A)$ are the singular values of A, for example, $\Delta_n(A)$ is the sum of eigenvalues of A, which is not necessarily equal to $s_1(A) + \cdots + s_n(A)$. One can also construct an example to show $\Delta_k(A^{-1}) \neq \Delta_k(A)^{-1}$.

3. Schur-Horn type result and Cheung-Tsing type result for $\Delta_k(A)$

If $A \in \mathbb{C}_{n\times n}$, then $\operatorname{diag} A$ denotes the diagonal matrix who diagonal is the diagonal of A. If $\lambda \in \mathbb{C}^n$, then $\operatorname{diag}(\lambda)$ denotes the diagonal matrix with λ as the diagonal. Let A be a Hermitian matrix with eigenvalues $\lambda \in \mathbb{R}^n$. Schur proved that the diagonal of A is majorized by λ and Horn established the converse [15]. So

$$\begin{aligned} \operatorname{diag}\{U\operatorname{diag}(\lambda_1,\dots,\lambda_n)U^{-1} : U \in U(n)\} &= \operatorname{conv}\{\operatorname{diag}(\lambda_\sigma), \sigma \in S_n\} \\ &= \{\operatorname{diag}(d) : d \in \operatorname{conv} S_n\lambda\}, \end{aligned} \tag{3.1}$$

where $\lambda_\sigma = (\lambda_{\sigma(1)}, \dots, \lambda_{\sigma(n)})$, "conv" denotes the convex hull of the underlying set, and $S_n\lambda$ denotes the orbit of λ under the action of the full symmetric group S_n.

A set S in a vector space is star shaped with star center c if the line segment joining c and any point $p \in S$ remains in S.

Theorem 3.1. (1) *For any $A \in \mathbb{C}_{n\times n}$, $\operatorname{diag}\Delta_k(A) = \Delta_k(\operatorname{diag}(A))$.*

(2) *Let $A \in \mathbb{C}_{n\times n}$ be Hermitian, $1 \leq k \leq n$. Then*

$$\begin{aligned} \operatorname{diag}(\Delta_k(UAU^{-1}) : U \in U(n)\} &= \operatorname{conv}\{\Delta_k(\operatorname{diag}(\lambda_\sigma)) : \sigma \in S_n\} \\ &= \operatorname{conv}\{\operatorname{diag}(\hat\lambda_\sigma) : \sigma \in S_n\}, \end{aligned}$$

where $\hat\lambda_\sigma = (\sum_{i=1}^k \lambda_{\sigma\alpha(i)})_{\alpha\in Q_{k,n}}$.

(3) *Let $A \in \mathbb{C}_{n\times n}$ be arbitrary, $1 \leq k \leq n$. Then $\{\operatorname{diag}(\Delta_k(UAU^{-1}) : U \in U(n)\}$ is star shaped with star center $\frac{k \operatorname{tr} A}{n} I_{\binom{n}{k}}$.*

Proof. (1) One can deduce from Fiedler's explicit formula of $\Delta_k(A)$, or using (2.2), or simply observe that the diagonal of $\Delta_k(A)$ is independent of the off diagonal entries of A.

(2) Notice that

$$\begin{aligned} \{\operatorname{diag}\Delta_k(UAU^{-1}) : U \in U(n)\} &= \{\Delta_k(\operatorname{diag} UAU^{-1}) : U \in U(n)\} \\ &= \{\Delta_k(\operatorname{diag}(d)) : d \in \operatorname{conv} S_n\lambda\} \end{aligned}$$

by (3.1). Since Δ_k is a linear map, it maps $\operatorname{conv}\operatorname{diag}(S_n\lambda) = \operatorname{diag}(\operatorname{conv} S_n\lambda)$ onto the convex hull of the points $(\Delta_k(\operatorname{diag}(\lambda_\sigma)), \sigma \in S_n$.

(3) Clearly $\Delta_k(I_n) = kI_{\binom{n}{k}}$. Then use the approach in (2) and a result of Cheung and Tsing [2] which asserts that the set $\{\operatorname{diag} UAU^{-1} : U \in U(n)\}$ is star shaped with star center $\frac{\operatorname{tr} A}{n} I_n$.

□

Remark 3.2. Schur-Horn's result is often stated in terms of majorization. Schur's part is equivilant to Fan's maximization principle [15]: $\max \sum_{i=1}^k (Ax_i, x_i) = \sum_{i=1}^k \lambda_i$ if $\lambda_1 \geq \cdots \geq \lambda_n$ are eigenvalues of the Hermitian A, where the maximum is taken over all sets of k orthonormal vectors $x_1, \dots, x_k$. London [12] gave a proof of Fan's principle via Δ_k since the diagonal entries of $\Delta_k(A)$ are of the form $\sum_{i=1}^k a_{\alpha(i),\alpha(i)}$, $\alpha \in Q_{k,n}$. It is in the same spirit of the standard proof via $C_k(A)$ [15, p.232], [1, p.43] of Weyl's inequalities for the singular values and eigenvalues of a complex matrix A.

4. Differential of induced operator in symmetry class of tensors

The following multilinear framework [13, 17, 21] enable us to extend the notion of additive compound and the results in the previous sections. Let V be an n-dimensional inner product space. Each element $\sigma \in S_k$ gives rise to a linear operator $P(\sigma)$ on $\otimes^k V$:

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_k) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}, \qquad v_1, \ldots, v_k \in V$$

on the decomposable tensors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ and then extended linearly to all of $\otimes^k V$. The map $\sigma \mapsto P(\sigma)$ is a unitary representation of S_k in $\otimes^k V$, that is, $P(\sigma_1)P(\sigma_2) = P(\sigma_1\sigma_2)$ and $P(\sigma)^{-1} = P(\sigma^{-1}) = P(\sigma)^*$.

Suppose H is a subgroup of S_k, and $\chi : H \to \mathbb{C}$ is a character of H (not necessarily of degree 1). The symmetrizer on the tensor space $\otimes^k V$,

$$S_\chi := \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma), \tag{4.1}$$

defined by H and χ, where e denotes the identity element of H, and $|H|$ is the order of H, is an orthoprojector. The range of S_χ,

$$V_\chi^k(H) := S_\chi(\otimes^k V),$$

is a subspace of $\otimes^k V$, called the symmetry class of tensors over V associated with H and χ. The elements in $V_\chi^k(H)$ of the form $S_\chi(v_1 \otimes \cdots \otimes v_k)$ are called decomposable symmetrized tensors and are denoted by $v_1 * \cdots * v_k$.

Let $I(H)$ be the set of irreducible characters of H. If $\chi, \xi \in I(H)$ and $\chi \neq \xi$, then $S_\chi S_\xi = 0$. Moreover $\sum_{\chi \in I(H)} \circ S_\chi = I$. So

$$\otimes^k V = \dot{\sum}_{\chi \in I(G)} V_\chi^k(H),$$

an orthogonal summand.

For any linear operator A acting on V, there is a (unique) induced operator $K(A)$ acting on $V_\chi^k(H)$ satisfying

$$K(A) v_1 * \cdots * v_k = A v_1 * \cdots * A v_k.$$

Indeed $V_\chi^k(H)$ is stable under $\otimes^k V$ and $K(A) = \otimes^k A|_{V_\chi^k(H)}$. Thus $K(A)v^* = (\otimes^k A)v^*$, $v^* \in V_\chi^k(H)$.

The induced inner product of $V_\chi^k(H)$ is given by

$$(x^*, y^*) = \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) \prod_{i=1}^k (x_t, y_{\sigma(i)}), \qquad x^*, y^* \in V_\chi^k(H).$$

When χ is linear, i.e., the degree of χ is 1, one can define the induced matrix $K(M)$ of a matrix M [17, p.235] so that if M is the matrix representation of A with respect to an orthonormal base, then $K(M)$ is the matrix representation of $K(A)$ with respect to the induced base. However, when χ is *not* linear, no such desirable induced matrix exists.

Since K is the restriction of $\otimes^k$ on $V_\chi^k(H)$ and $\otimes^k : GL_n(\mathbb{C}) \to \mathrm{Aut}\,(\otimes^k \mathbb{C}^n)$ is continuous, K is continuous. Here we view $GL_n(\mathbb{C})$ as the group of invertible operator on $\mathbb{C}^n$. Since $\otimes^k A \otimes^k B = \otimes^k(AB)$, we have $K(AB) = K(A)K(B)$ [13, p.185]. So $K : GL_n(\mathbb{C}) \to \mathrm{Aut}\,(V_\chi^n(H))$ is a continuous thus analytic representation. Denote by $dK : \mathfrak{gl}_n(\mathbb{C}) \to \mathrm{End}\,(V_\chi^n(H))$ the differential of K at the identity and it is given by

$$dK(A) = \frac{d}{dt}\Big|_{t=0} (K(e^{tA})),$$

which is also obtained via $K(I + tA) = I + t\, dK(A) + t^2 R + \cdots$.

Remark 4.1. It is known that if $\chi(e) > 1$, then K (and thus dK) is always reducible. When χ is a linear character, Merris [16] gave a criterion for the irreduciblity of K.

Example 4.2.
(1) When $H = \{e\} < S_k$ so that $\chi \equiv 1$, $V_\chi^k(H) = \otimes^k V$ and $K(A) = \otimes^k A$ and $dK(A) = A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A$.
(2) When $H = S_k$ $(1 \le k \le n)$ and $\chi(\sigma) = \epsilon(\sigma)$, i.e., the alternating character, $V_\chi^k(H) = \wedge^k V$, $K(A) = C_k(A)$ and $dK(A) = \Delta_k(A)$.
(3) When $H = S_k$ $(1 \le k \le n)$ and $\chi \equiv 1$, $V_\chi^k(H) = \vee^k V$, the completely symmetric space, $K(A) = P_k(A)$ and $dK(A) = Q_k(A)$ in [12, p.187].

In order to compute $dK(A)$, we follow the idea in [12]

$$\begin{aligned}
dK(A)v^* &= \frac{d}{dt}\Big|_{t=0} K(e^t A)\\
&= \frac{d}{dt}\Big|_{t=0} K(I + tA)\\
&= \lim_{t\to 0} \frac{(I + tA)v_1 * \cdots * (I + tA)v_k - v_1 * \cdots * v_k}{t}\\
&= \lim_{t\to 0} \frac{(v_1 + tAv_1) * \cdots * (v_k + tAv_p) - v_1 * \cdots * v_k}{t}\\
&= \sum_{i=1}^{k} v_1 * \cdots * Av_i * \cdots * v_k. \qquad (4.2)
\end{aligned}$$

Proposition 4.3. *Let* $D_e(A) := A \otimes I \otimes \cdots \otimes I + I \otimes A \otimes I \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes A$*. For each* $\chi \in I(H)$*,* $V_\chi^k(H)$ *is an invariant subspace of* $D_e(A)$ *and* $dK(A) = D_e(A)\,\big|_{V_\chi^k(H)}$.

Proof. Notice that D_e is the derivative in Example 4.2 and commutes with $P(\sigma)$ and thus commute with S_χ. So $V_\chi^k(H)$ is an invariant subspace of $D_e(A)$. Now $D_e(A)\,\big|_{V_\chi^k(H)}$, the restriction of $D_e(A)$ on $V_\chi^k(H)$, is $dK(A)$ since by (4.2)

$$\begin{aligned}
dK(A)v^* &= Av_1 * v_2 * \cdots * v_k + v_1 * Av_2 * v_3 * \cdots * v_k + \cdots\\
&\quad + v_1 * \cdots * v_{k-1} * Av_k\\
&= S_\chi(Av_1 \otimes v_2 \otimes \cdots \otimes v_k + v_1 \otimes Av_2 \otimes v_3 \otimes \cdots \otimes v_k + \cdots\\
&\quad + v_1 \otimes \cdots \otimes v_{k-1} \otimes Av_k)\\
&= S_\chi D_e(A) v^\otimes\\
&= D_e(A) v^*
\end{aligned}$$

□

Remark 4.4. The results in the previous sections can be extended for $dK(A)$ and the details are left to the interested readers. When Schur-Horn's extension and Cheung-Tsing's extension are considered, some caution has to be made since we do not have a desirable definition of induced matrix [17, p.238], and thus no desirable "differential matrix", to work with when χ is not linear. However it can still be dealt with as we will see in Remark 5.6.

5. Further generalization in the context of reductive Lie group

We want to discuss the possible extensions of the results in the previous sections in the context of Lie group since the general linear group $GL_n(\mathbb{C})$ is a Lie group. A representation π of G on the (finite dimensional complex) vector space V_π is a continuous (and thus analytic [6, p.117]) map $\pi : G \to \mathrm{Aut}\,(V_\pi)$ such that $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ for all $g_1, g_2 \in G$. The following is an immediate consequence of [6, p.110]) and is evidently a generalization of Proposition 2.1.

Proposition 5.1. *Let $\pi : G \to \mathrm{Aut}\,(V_\pi)$ be a representation of the Lie group G. Then the differential of π at the identity, $d\pi : \mathfrak{g} \to \mathrm{End}\,(V_\pi)$, is a Lie algebra homomorphism, i.e., $d\pi[A,B] = [d\pi(A), d\pi(B)]$ and $\pi \circ \exp = \exp \circ\, d\pi$.*

Let G be a reductive Lie group. For us, a reductive Lie group is a member of the so-called Harish-Chandra class [10, p.384]. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of $\mathfrak{g}$. Let $\pi : G \to \mathrm{Aut}\,(V_\pi)$ be a representation of G and let $d\pi : \mathfrak{g} \to \mathrm{End}\,(V_\pi)$ be the differential of π at the identity. The Lie bracket $[X, Y] = XY - YX$, $X, Y \in \mathfrak{gl}_n(\mathbb{C})$. Moreover, $A \in \mathfrak{gl}_n(\mathbb{C})$ is normal if and only if $AA^* - A^*A = 0$, i.e., $[A, A^*] = 0$. The following is a generalization of Proposition 2.3.

Proposition 5.2. *Let $\pi : G \to \mathrm{Aut}\,(V_\pi)$ be a representation of the reductive Lie group G. If $A, B \in \mathfrak{g}$, then*

1. *$d\pi(A) : V_\pi \to V_\pi$ is linear.*
2. *$d\pi(\mathrm{Ad}(g)A) = \mathrm{Ad}(\pi(g))d\pi(A)$.*
3. *$d\pi[A,B] = [d\pi(A), d\pi(B)]$.*
4. *If $G = GL_n(\mathbb{C})$, and suppose that $d\pi(I_n) = \alpha I_{V_\pi}$, $\alpha \in \mathbb{R}$, then there is an inner product structure on V_π such that for Hermitian A, $d\pi(A)$ is Hermitian and for skew-Hermitian A, $d\pi(A)$ is skew Hermitian.*
5. *With respect to the inner product in (4), if $A \in \mathfrak{gl}_n(\mathbb{C})$ is normal, then $d\pi(A)$ is normal.*
6. *If $G = GL_n(\mathbb{C})$, and suppose that $d\pi(I_n) = \alpha I_{V_\pi}$, $\alpha \in \mathbb{C}$, then the eigenvalues of $d\pi(A) \in \mathrm{End}\,(V_\pi)$ are the eigenvalues of $d\pi(diag\,(\lambda_1, \dots, \lambda_n))$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of $A \in \mathfrak{gl}_n(\mathbb{C})$.*

Proof. The first and the third parts follow from Proposition 2.3.

(2) Consider the representation $\pi \circ i_g : G \to \mathrm{Aut}\,(V_\pi)$ where $i_g : G \to G$ is the automorphism $i_g(h) = ghg^{-1}$. By Proposition 5.1,

$$e^{td(\pi\circ i_g)(A)} = \pi \circ i_g(e^{tA}) = \pi(ge^{tA}g^{-1}) = \pi(g)\pi(e^{tA})\pi(g^{-1}) = \mathrm{Ad}(\pi(g))\pi(e^{tA}).$$

Thus

$$d(\pi \circ i_g)(A) = \frac{d}{dt}\Big|_{t=0} \pi \circ i_g(e^{tA}) = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}(\pi(g))\pi(e^{tA}) = \mathrm{Ad}(\pi(g))d\pi(A).$$

(4) Notice that $\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{C}) + \mathfrak{z}$ where $\mathfrak{z} = \{cI_n : c \in \mathbb{C}\}$ is the center of $\mathfrak{gl}_n(\mathbb{C})$ and $\mathfrak{sl}_n(\mathbb{C})$ is semisimple. Let $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}_n + i\mathfrak{su}_n$ be the Cartan decomposition of

$\mathfrak{sl}_n(\mathbb{C})$, where $\mathfrak{su}_n$ is the Lie algebra of the special unitary group $SU(n)$ which is simply connected. So there exists [21, p.101] a unique homomorphism $\hat{\pi} : SU(n) \to \mathrm{Aut}\,(V_\pi)$ such that $d\hat{\pi} = d\pi$. There exists an inner product $(\cdot,\cdot)$ on V_π such that $\hat{\pi}(u)$ is orthogonal for all $u \in SU(n)$: starting with any arbitrary inner product $\langle\cdot,\cdot\rangle$ on V_π, set

$$(x,y) = \int_{SU(n)} \langle \hat{\pi}(g)x, \hat{\pi}(g)y\rangle \, dg,$$

where the integral is normalized and left-invariant. Differentiate the identity

$$(\hat{\pi}(e^{tZ})X, \hat{\pi}(e^{tZ})Y) = (X,Y)$$

for all $X, Y \in V_\pi$ at $t = 0$ we have

$$(d\pi(Z)X, Y) = -(X, d\pi(Z)Y).$$

Thus with respect to the $\hat{\pi}(SU(n))$ invariant inner product, $d\pi(Z)$ is skew Hermitian for all $Z \in \mathfrak{su}_n$. Then $d\pi(Z)$ is skew Hermitian if $Z \in \mathfrak{su}_n$ and is Hermitian if $Z \in i\mathfrak{su}_n$. By translation and the assumption $d\pi(I_n) = \alpha I_{V_\pi}$ for some $\alpha \in \mathbb{R}$, we have the desired result.

(5) $A \in \mathfrak{gl}_n(\mathbb{C})$ is normal if and only if $[A, A^*] = 0$. By (3), $[d\pi(A), d\pi(A^*)] = 0$. By (4), $d\pi(A^*) = (d\pi(A))^*$ so that $d\pi(A)$ is normal.

(6) We first consider $A \in \mathfrak{sl}_n(\mathbb{C}) \subset \mathfrak{gl}_n(\mathbb{C})$ since $d\pi$ is linear. By the Jordan decomposition, $A = A_s + A_n$ with $[A_s, A_n] = 0$, where $A_s, A_n \in \mathfrak{sl}_n(\mathbb{C})$ are the (unique) semisimple part (i.e., A, or equivalently, $\mathrm{ad}\,(A)$ is diagonalizable over $\mathbb{C}$) and the (unique) nilpotent part of A, respectively [9, p.29]. Now $d\pi(A) = (d\pi(A))_s + (d\pi(A))_n$, and [9, p.30] $d\pi(A_s) = (d\pi(A))_s$ and $d\pi(A_n) = (d\pi(A))_n$. Notice that $\mathrm{Ad}(g)A_s = gA_sg^{-1} = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)$ for some $g \in SL_n(\mathbb{C})$ since A_s is diagonalizable. So the eigenvalues of $d\pi(A_s)$ and thus of $d\pi(A)$ are those of

$$\pi(g)d\pi(A_s)\pi(g)^{-1} = \mathrm{Ad}(\pi(g))d\pi(A_s) = d\pi(\mathrm{Ad}(g)A_s) = d\pi(\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n)).$$

For the general case, each $A' \in \mathfrak{gl}_n(\mathbb{C})$ can be rewritten $A' = A + aI_n$ where $a = \frac{\mathrm{tr}\, A'}{n}$ and $A \in \mathfrak{sl}_n(\mathbb{C})$. Since $d\pi(I_n) = \alpha I_{V_\pi}$, $d\pi(A_s + aI_n)$ is diagonalizable and commutes with the nilpotent part $d\pi(A_n)$, the eigenvalues of $d\pi(A')$ are those of $d\pi(A_s + aI_n)$. If $g(A_s + aI_n)g^{-1} = \mathrm{diag}\,(\lambda_n, \ldots, \lambda_n)$, where λ's are the eigenvalues of A', then $\pi(g)d\pi(A_s + aI_n)\pi(g)^{-1} = d\pi(\mathrm{diag}\,(\lambda_1, \ldots, \lambda_n))$ and we have the desired result.

□

Remark 5.3. The first three parts of Proposition 5.2 hold for general Lie group, i.e., not necessarily reductive.

Remark 5.4. Let $G = GL_n(\mathbb{C})$. Note that $\Delta_k(I_n) = kI_{\binom{n}{k}}$ and it holds for dK as well. If $V_\pi = \mathfrak{gl}_n(\mathbb{C})$ and $\pi = \mathrm{Ad}$, then $d\pi = \mathrm{ad}$ and clearly $\mathrm{ad}\,(I_n) = 0$. However $d\pi(I_n)$ may not be a scalar multiple of the identity in $\mathrm{End}\,(V_\pi)$. Consider the representation

$$\pi : A \mapsto A \oplus \mathbb{C}_k(A), \qquad V_\pi = \mathbb{C}^n \oplus \mathbb{C}^{\binom{n}{k}},$$

so that $d\pi(A) = A \oplus \Delta_k(A)$. Hence $d\pi(I_n) = I_n \oplus kI_{\binom{n}{k}}$ and is not a scalar if $k \neq 1, n-1$.

Now we turn to the extensions of the results of Schur-Horn and Cheung-Tsing. However we do not have an explicit form of $d\pi$ like the additive compound $\Delta_k(A)$ to work with. Let G be a reductive group and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Fix a maximal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$. Let $\xi : \mathfrak{p} \to \mathfrak{a}$ denote the orthogonal projection with respect to the Killing form and let $\mathfrak{a}^\perp$ denote the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. Set $\mathfrak{g}' := d\pi(\mathfrak{g})$, $\mathfrak{k}' := d\pi(\mathfrak{k})$, $\mathfrak{p}' := d\pi(\mathfrak{p})$, $\mathfrak{a}' := d\pi(\mathfrak{a})$ so that we have $\mathfrak{g}' = \mathfrak{k}' + \mathfrak{p}'$ for the Lie algebra $\mathfrak{g}'$. Let $\xi' : \mathfrak{p}' \to \mathfrak{a}'$ be the orthogonal projection of $\mathfrak{p}'$ onto $\mathfrak{a}'$ with respect to the Killing form $B(\cdot, \cdot)$ on $\mathrm{End}\,(V_\pi)$. The following is an extension of Theorem 3.1, i.e., the linear projection diag in Theorem 3.1 will be replaced by ξ and ξ'.

Theorem 5.5. *Let $\pi : G \to \mathrm{Aut}\,(V_\pi)$ be a representation of the reductive Lie group G.*

1. *For all $A \in \mathfrak{p}$, $\xi' \circ d\pi(A) = d\pi \circ \xi(A)$.*
2. *If $A \in \mathfrak{a}$, then $\xi' \circ d\pi(\mathrm{Ad}(K)A) = \mathrm{conv}\, d\pi(WA) = \mathrm{conv}\, \pi(W)d\pi(A)$, where W is the Weyl group of $(\mathfrak{a}, \mathfrak{g})$ and conv denotes the convex hull of the underlying set.*
3. *Suppose $G = GL_n(\mathbb{C})$ and let $A \in \mathfrak{gl}_n(\mathbb{C})$ be arbitrary. Then $\{\xi' \circ d\pi(UAU^{-1}) : U \in U(n)\}$ is star shaped with star center $\frac{\mathrm{tr}\, A}{n} d\pi(I_n)$.*

Proof. (1) It is known that [7] $\mathfrak{a}^\perp = [\mathfrak{k}, \mathfrak{a}]$. So $d\pi(\mathfrak{a}^\perp) = d\pi[\mathfrak{k}, \mathfrak{a}] = [d\pi(\mathfrak{k}), d\pi(\mathfrak{a})]$. Let $B(\cdot, \cdot)$ denote the Killing form on $\mathrm{End}\,(V_\pi)$. Then for any $A \in \mathfrak{a}$ and $B = [X, Y] \in \mathfrak{a}^\perp$ where $X \in \mathfrak{k}$ and $Y \in \mathfrak{a}$, using $B(U, [V, W]) = B(V, [W, U])$, $U, V, W \in \mathfrak{g}$ [6, p.131],

$$\begin{aligned} B(d\pi(A), d\pi[X, Y]) &= B(d\pi(A), [d\pi(X), d\pi(Y)]) \\ &= B(d\pi(X), [d\pi(Y), d\pi(A)]) \\ &= B(d\pi(X), d\pi[Y, A]) \\ &= 0. \end{aligned}$$

So $d\pi(\mathfrak{a}^\perp) \subset (d\pi(\mathfrak{a}))^\perp$, the orthogonal complement of $\mathfrak{a}'$ in $\mathfrak{p}'$. Since $\dim(d\pi(\mathfrak{a}) \cap d\pi(\mathfrak{a}^\perp)) = \dim(\mathfrak{a} \cap \mathfrak{a}^\perp) = 0$, we have $\dim d\pi(\mathfrak{a}) + \dim d\pi(\mathfrak{a}^\perp) = \dim \mathfrak{p}$ and thus $d\pi(\mathfrak{a}^\perp) = (d\pi(\mathfrak{a}))^\perp$. Actually the statement and the proof remain true if $d\pi$ is replaced by any Lie algebra homomorphism $\tau : \mathfrak{g} \to \mathrm{End}\,(V_\pi)$.

(2) By the first part, $\xi' \circ d\pi(\mathrm{Ad}(K)A) = d\pi \circ \xi(\mathrm{Ad}(K)A)$. By Kostant linear convexity theorem [7, Theorem 3.6] which asserts that $\xi(\mathrm{Ad}(K)A) = \mathrm{conv}\, WA$, we have

$$\xi' \circ d\pi(\mathrm{Ad}(K)A) = d\pi \circ \xi(\mathrm{Ad}(K)A) = d\pi(\mathrm{conv}\, WA) = \mathrm{conv}\, d\pi(WA),$$

since $d\pi$ is linear. By using Proposition 5.2 (2), we also have

$$\xi' \circ d\pi(\mathrm{Ad}(K)A) = \mathrm{conv}\, \pi(W)d\pi(A).$$

(3) Same idea as in the proof of Theorem 3.1. □

Remark 5.6. (1) Let $G = GL_n(\mathbb{C})$. Notice that $\{E_{11}, \ldots, E_{nn}\}$, where E_{ii} denotes the diagonal matrix having 1 as its ith diagonal entry and zero elsewhere, is a basis for $\mathfrak{a}$, the algebra of real diagonal matrices in the space of Hermitian matrices $\mathfrak{p} \subset \mathbb{C}_{n\times n}$. Though $\dim p = n^2 < \binom{n}{k}^2$ (when $n \neq 1, n-1, n$) which is the dimension of the space of Hermitian elements in $\mathrm{End}\,(\wedge^k(\mathbb{C}^n))$, the elements

$$E_{\alpha,\alpha} := \Delta_k(\sum_{i=1}^{k} E_{\alpha(i),\alpha(i)}) \in \mathrm{End}\,(\wedge^k(\mathbb{C}^n)), \qquad \alpha \in Q_{k,n},$$

form a basis for the algebra of diagonal matrices in $\mathrm{End}\,(\wedge^k(\mathbb{C}^n))$. So we have Theorem 3.1 as a corollary of Theorem 5.5.

(2) For symmetry classes of tensors $V_\chi^k(H)$,

$$E_{\gamma,\gamma} := dK(\sum_{i=1}^{k} E_{\gamma(i),\gamma(i)}) \in \mathrm{End}\,(V_\chi^k(H)), \qquad \gamma \in \hat{\Delta},$$

form a basis for the diagonal elements in $\mathrm{End}\,(V_\chi^k(H))$ with respect to the basis $E^* = \{e_\gamma^* : \gamma \in \hat{\Delta}\}$ (see the notations $\hat{\Delta}$ in [17, p.238] and the proof of [17, Lemma 7.48]), where $E = \{e_1, \ldots, e_n\}$ denotes the standard basis of $\mathbb{C}^n$. Of course, when χ is linear, $\hat{\Delta}$ is an orthogonal basis and induced matrix (and thus the matrix of dK) is defined. The situation is similar to the exterior case.

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