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# DETERMINANT AND PFAFFIAN OF SUM OF SKEW SYMMETRIC MATRICES

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Abstract. We completely describe the determinants of the sum of orbits of two real skew symmetric matrices, under similarity action of orthogonal group and the special orthogonal group respectively. We also study the Pfaffian case and the complex case.

## 1. Introduction

Let $\mathrm{U}(n) \subset \mathbb{C}_{n\times n}$ be the unitary group. Fiedler [3] obtained the following result.

**Theorem 1.1.** (Fiedler) Let $A, B \in \mathbb{C}_{n\times n}$ be Hermitian matrices with eigenvalues $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$. Then

$$H(A,B) := \{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{U}(n)\}$$

is the interval $[\min_{\sigma\in S_n} \prod_{i=1}^n (\alpha_i+\beta_{\sigma(i)}), \max_{\sigma\in S_n} \prod_{i=1}^n (\alpha_i+\beta_{\sigma(i)})]$, where $S_n$ denotes the symmetric group on $\{1, \ldots, n\}$.

See [2, 6, 7] for related studies. The result remains the same if $\mathrm{U}(n)$ is replaced by the special unitary group $\mathrm{SU}(n)$. It is also the same if $A, B \in \mathbb{R}_{n\times n}$ are real symmetric matrices and $\mathrm{U}(n)$ is replaced by the orthogonal group $\mathrm{O}(n)$ or the spectral orthogonal group $\mathrm{SO}(n)$.

If $A, B \in \mathbb{R}_{n\times n}$ are skew symmetric, we consider the set

$$\begin{aligned} D(A,B) &:= \{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{O}(n)\} \\ &= \{\det(A + VBV^{-1}) : V \in \mathrm{O}(n)\}. \end{aligned}$$

The case $n = 2k+1$ is trivial since $n \times n$ real skew symmetric matrices are singular so that $D(A,B) = \{0\}$. We only need to consider the even case $n = 2k$. Theorem 1.1 can only provide lower and upper bounds for the compact set $D(A,B) \subset \mathbb{R}$ via the Hermitian matrices $iA$ and $iB$. The orthogonal group $\mathrm{O}(n)$ is not connected and has two

2000 Mathematics Subject Classification. Primary 15A15, 15A42 components corresponding to $+1$ and $-1$ determinant, namely, the special orthogonal group $\mathrm{SO}(n)$ and $\hat{\mathrm{SO}}(n) := D\mathrm{SO}(n)$, where

$$D := \mathrm{diag}\,(1, \dots, 1, -1).$$

Though $\mathrm{O}(n)$ is disconnected, we show that $D(A, B)$ is an interval when $k \geq 2$. In Section 2 we determine $D(A, B)$ and

$$D_0(A, B) := \{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{SO}(n)\}.$$

In Section 3 we study the case when the determinant function is replaced by the Pfaffian. In Section 4 we obtain corresponding result for the complex case.

## 2. Determinant and sum of orbits

Let $A, B \in \mathbb{R}_{2k\times 2k}$ be skew symmetric. Denote by

$$\begin{aligned} \Delta(A, B) &:= \{UAU^{-1} + VBV^{-1} : U, V \in \mathrm{O}(2k)\} \\ \Delta_0(A, B) &:= \{UAU^{-1} + VBV^{-1} : U, V \in \mathrm{SO}(2k)\} \end{aligned}$$

the sums of orbits of $A$ and $B$ under $\mathrm{O}(2k)$ and $\mathrm{SO}(2k)$ respectively. Set

$$D(A, B) := \{\det M : M \in \Delta(A, B)\}$$

and

$$D_0(A, B) := \{\det M : M \in \Delta_0(A, B)\}.$$

Notice that $D_0(A, B)$ is a closed interval since $\mathrm{SO}(2k)$ is compact and connected.

The spectral theorem for real skew symmetric matrix $A$ under $\mathrm{O}(2k)$ [5] asserts that there is an orthogonal matrix $O \in \mathrm{O}(2k)$ such that $OAO^{-1}$ is in the following canonical form

$$OAO^{-1} = \alpha_1 J \oplus \cdots \oplus \alpha_k J = \mathrm{diag}\,(\alpha_1, \dots, \alpha_k) \otimes J \tag{1}$$

where

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Thus we may assume that $A$ and $B$ are in their canonical forms when $D(A, B)$ is studied. The eigenvalues of $A$ are purely imaginary and occur in pairs

$$\pm i\alpha_1, \dots, \pm i\alpha_k$$

and we may assume that $\alpha_1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \alpha_k (\geq 0)$ which are the singular values of $A$.

However the spectral theorem for $A$ under $\mathrm{SO}(2k)$ involves the sign of the Pfaffian of $A$, denoted by $\delta_A = \pm 1$. To quickly see it, consider $J$ and $-J$ which are similar via $\mathrm{O}(2)$ but not $\mathrm{SO}(2)$. Eigenvalues fail to distinguish $J$ and $-J$. However $\operatorname{Pf} J = 1$ and $\operatorname{Pf}(-J) = -1$, where Pfaffian of $A$ [4] is defined as

$$\operatorname{Pf} A = \frac{1}{2^k k!} \sum_{\sigma \in S_{2k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} a_{\sigma(2i-1),\sigma(2i)},$$

and $\operatorname{sgn}(\sigma)$ is the signature of $\sigma$. For example

$$\operatorname{Pf} \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} = a, \quad \operatorname{Pf} \begin{bmatrix} 0 & a & b & c \\ -a & 0 & d & e \\ -b & -d & 0 & f \\ -c & -e & -f & 0 \end{bmatrix} = af - be + dc.$$

In particular, if $A = \operatorname{diag}(a_1, \ldots, a_k) \otimes J$, then $\operatorname{Pf} A = a_1 \cdots a_k$. It is well known that [8, 9]

$$(\operatorname{Pf} A)^2 = \det A.$$

If $A \in \mathbb{R}_{2k\times 2k}$ is skew symmetric, there is $O \in \mathrm{SO}(2k)$ such that

$$OAO^{-1} = \begin{cases} \operatorname{diag}(\alpha_1, \ldots, \alpha_k) \otimes J & \text{if } \delta_A = 1 \\ \operatorname{diag}(\alpha_1, \ldots, -\alpha_k) \otimes J & \text{if } \delta_A = -1 \end{cases} \tag{2}$$

**Lemma 2.1.** Let $A, B \in \mathbb{R}_{2k\times 2k}$ be real skew symmetric. Then

(a) $D_0(A, B) \subseteq \mathbb{R}_+$.
(b) $D(A, B) = D_0(A, B) \cup D_0(A_D, B)$, where $A_D := DAD^{-1}$.

*Proof.* (a) The determinant of each real skew symmetric matrix in $\mathbb{R}_{2k\times 2k}$ is nonnegative so $D_0(A, B) \subseteq \mathbb{R}_+$.
(b) It is easy to see that

$$\Delta(A, B) = \Delta_0(A, B) \cup \Delta_0(A_D, B) \cup \Delta_0(A, B_D) \cup \Delta_0(A_D, B_D)$$

where $A_D := DAD^{-1}$. Notice that $D_0(A, B) = D_0(A_D, B_D)$ and $D_0(A_D, B) = D_0(A, B_D)$ since

$$\begin{aligned} &= D_0(A, B) \\ &= \{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{SO}(2k)\} \\ &= \{\det(DUD^{-1}A_D DU^{-1}D^{-1} + DVD^{-1}B_D DV^{-1}D^{-1}) : U, V \in \mathrm{SO}(2k)\} \\ &= D_0(A_D, B_D). \end{aligned}$$

□

Notice that $D_0(A, B)$ and $D_0(A, B_D)$ are closed intervals. We will show that is the case for $D(A, B)$ when $k \geq 2$.

**Lemma 2.2.** Let $k \geq 2$ and let $A, B \in \mathbb{R}_{2k\times 2k}$ be real skew symmetric with singular values $\alpha_1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \alpha_k$ and $\beta_1 \geq \beta_1 \geq \beta_2 \geq \beta_2 \geq \cdots \geq \beta_k \geq \beta_k$. The following are equivalent.

(a) $\Delta(A,B) \subseteq \mathrm{GL}_{2k}(\mathbb{R})$.
(b) $\Delta_0(A,B) \subseteq \mathrm{GL}_{2k}(\mathbb{R})$.
(c) $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$.

*Proof.* (a) ⇒ (b) is obvious.

(b) ⇒ (c) Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$, i.e., either (i) $\alpha_k \le \beta_k \le \alpha_1$ or (ii) $\beta_k \le \alpha_k \le \beta_1$. By symmetry we may assume that (i) holds. Because of (2) and $D_0(A,B) = D_0(A_D, B_D)$ we may assume that

$$A = \mathrm{diag}\,(\alpha_1, \dots, \alpha_k) \otimes J, \quad B = \mathrm{diag}\,(\beta_1, \dots, \delta\beta_k) \otimes J$$

where $\delta := \delta_A\delta_B = \pm 1$. Set $h(V) := \mathrm{Pf}\,(A + VBV^{-1})$, $V \in \mathrm{SO}(2k)$. Clearly

$$h(I) = \begin{cases} \prod_{i=1}^k (\alpha_i + \beta_i) \ge 0 & \text{if } \delta = 1 \\ (\alpha_k - \beta_k) \prod_{i=1}^{k-1} (\alpha_i + \beta_i) \le 0 & \text{if } \delta = -1. \end{cases}$$

Now let $V \in \mathrm{SO}(2k)$ such that

$$VBV^{-1} = \mathrm{diag}\,(-\beta_k, \beta_2, \dots, \beta_{k-1}, -\delta\beta_1) \otimes J.$$

Then

$$h(V) = \begin{cases} (\alpha_1 - \beta_k)(\alpha_k - \beta_1) \prod_{i=2}^{k-1} (\alpha_i + \beta_i) \le 0 & \text{if } \delta = 1 \\ (\alpha_1 - \beta_k)(\alpha_k + \beta_1) \prod_{i=2}^{k-1} (\alpha_i + \beta_i) \ge 0 & \text{if } \delta = -1. \end{cases}$$

By the path connectedness of $\mathrm{SO}(2k)$ and the continuity of the Pfaffian, there is $W \in \mathrm{SO}(2k)$ such that $h(W) = \mathrm{Pf}\,(A + WBW^{-1}) = 0$, so $A + WBW^{-1}$ is singular.

(c) ⇒ (a) See [6, Lemma 3]. □

The next two theorems determine $D_0(A,B)$ and $D(A,B)$ respectively.

**Theorem 2.3.** Let $A, B \in \mathbb{R}_{2k\times 2k}$ be real skew symmetric with singular values $\alpha_1 \ge \alpha_1 \ge \alpha_2 \ge \alpha_2 \ge \cdots \ge \alpha_k \ge \alpha_k$ and $\beta_1 \ge \beta_1 \ge \beta_2 \ge \beta_2 \ge \cdots \ge \beta_k \ge \beta_k$ respectively. Let $\delta_A$ and $\delta_B$ be the signs of the Pfaffians of $A$ and $B$, respectively. Let $D_0(A,B) = [m_0, M_0]$.
(a) If $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$, then

$$M_0 = \begin{cases} \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2 & \text{if } \delta_A\delta_B = 1 \\ \max_{1\le j\le k} (\alpha_j - \beta_{k-j+1})^2 \prod_{i\neq j}^k (\alpha_i + \beta_{k-i+1})^2 & \text{if } \delta_A\delta_B = -1 \end{cases}$$

and

$$m_0 = \begin{cases} \prod_{i=1}^k (\alpha_i - \beta_{k-i+1})^2 & \text{if } \quad \delta_A\delta_B = 1, k \text{ even} \\ & \text{or } \quad \delta_A\delta_B = -1, k \text{ odd} \\ \min_{1\le j\le k} (\alpha_j + \beta_{k-j+1})^2 \prod_{i\neq j}^k (\alpha_i - \beta_{k-i+1})^2 & \text{if } \quad \delta_A\delta_B = -1, k \text{ even} \\ & \text{or } \quad \delta_A\delta_B = 1, k \text{ odd} \end{cases}$$

(b) If $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$, we have two cases:
(I) if $k = 1$, then

$$D_0(A, B) = \begin{cases} 4\alpha_1^2 & \text{if } \delta_A \delta_B = 1 \\ 0 & \text{if } \delta_A \delta_B = -1. \end{cases}$$

(II) if $k \geq 2$, then $D_0(A, B) = [0, M_0]$.

*Proof.* We consider two cases.
(I) If $k = 1$, then $D_0(A, B) = \{s\}$, where

$$s = \begin{cases} (\alpha_1 + \beta_1)^2 & \text{if } \delta_A \delta_B = 1 \\ (\alpha_1 - \beta_1)^2 & \text{if } \delta_A \delta_B = -1 \end{cases}$$

(II) Let $k \geq 2$.
(a) Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$. Because of (2) and $D_0(A, B) = D_0(A_D, B_D)$ we may assume that

(3) $$A = \operatorname{diag}(\alpha_1, \ldots, \alpha_k) \otimes J, \quad B = \operatorname{diag}(\beta_1, \ldots, \beta_{k-1}, \delta\beta_k) \otimes J$$

where $\delta := \delta_A \delta_B$. Notice that $m_0$ and $M_0$ depend continuously on $\alpha_1 \geq \cdots \geq \alpha_k$. So we may assume that

(4) $$\alpha_1 > \alpha_2 > \cdots > \alpha_k > 0.$$

By Lemma 2.2 all matrices in $D_0(A, B)$ are nonsingular. Assume that

$$D_0(A, B) = \{\det(A + VBV^{-1}) : V \in \mathrm{SO}(2k)\}$$

attains its minimum $m_0$ or maximum $M_0$ at $\det(A + B_0)$, where

$$B_0 = V_0 B V_0^{-1}$$

for some $V_0 \in \mathrm{SO}(2k)$. The optimizing matrix

(5) $$C_0 := A + B_0$$

is nonsingular and $C_0^{-1}$ is skew symmetric. For each real skew symmetric matrix $S \in \mathbb{R}_{2k\times 2k}$, $e^{\epsilon S} \in \mathrm{SO}(2k)$. Recall Fiedler's lemma [3]:

$$\det(P + \epsilon Q) = (\det P)(1 + \epsilon \operatorname{tr} QP^{-1}) + o(\epsilon^2),$$

where $P, Q \in \mathbb{C}_{n\times n}$ and $P$ is nonsingular. Hence

$$\begin{aligned} \det(A + e^{\epsilon S} B_0 e^{-\epsilon S}) &= \det(A + B_0 + \epsilon\,[S, B_0]) + o(\epsilon^2) \\ &= \det C_0 (1 + \epsilon \operatorname{tr}[S, B_0] C_0^{-1}) + o(\epsilon^2) \\ &= \det C_0 (1 + \epsilon \operatorname{tr} S[B_0, C_0^{-1}]) + o(\epsilon^2) \end{aligned}$$

so that $\operatorname{tr} S[B_0, C_0^{-1}] = 0$ for all real skew symmetric $S$. Choose $S = [B_0, C_0^{-1}]$ to have $[B_0, C_0^{-1}] = 0$, i.e., $B_0$ commutes with $C_0^{-1}$ and thus with $C_0$, i.e., $B_0(A+B_0)=(A+B_0)B_0$. Hence $B_0A=AB_0$. Because of (3) and (4),

$$B_0=\mathrm{diag}\,(\pm\beta_{\sigma(1)},\ldots,\pm\beta_{\sigma(k)})\otimes J$$

for some $\sigma\in S_n$, with

$$\text{the number of subtracted terms is}\begin{cases}\text{even} & \text{if } \delta=1\\ \text{odd} & \text{if } \delta=-1.\end{cases} \tag{6}$$

This becomes a finite optimization problem. The maximum and minimum occur among the numbers

$$\prod_{i=1}^{k}(\alpha_i\pm\beta_{\sigma(i)})^2,\quad \sigma\in S_k \tag{7}$$

subject to (6). The maximum is attained at

$$B_0=\mathrm{diag}\,(\beta_{\sigma(1)},\ldots,\beta_{\sigma(k-1)},\delta\beta_{\sigma(k)})\otimes J$$

for some $\sigma\in S_k$ and we have

$$M_0=\begin{cases}\max_{\sigma\in S_k}\prod_{i=1}^{k}(\alpha_i+\beta_{\sigma(i)})^2 & \text{if}\quad \delta=1\\ \max_{\substack{\sigma\in S_k\\ 1\le j\le k}}(\alpha_j-\beta_{\sigma(j)})^2\prod_{i\ne j}^{k}(\alpha_i+\beta_{\sigma(i)})^2 & \text{if}\quad \delta=-1.\end{cases}$$

The expression $\max_{\sigma\in S_k}\prod_{i=1}^{k}(\alpha_i+\beta_{\sigma(i)})^2$ can be identified. Notice that [3, p.29] for $i<j$ and $\sigma(i)<\sigma(j)$,

$$(\alpha_i+\beta_{\sigma(i)})(\alpha_j+\beta_{\sigma(j)})-(\alpha_i+\beta_{\sigma(j)})(\alpha_j+\beta_{\sigma(i)})=-(\alpha_i-\alpha_j)(\beta_{\sigma(i)}-\beta_{\sigma(j)})\le 0$$

for each $\sigma\in S_k$. So

$$\max_{\sigma\in S_k}\prod_{i=1}^{k}(\alpha_i+\beta_{\sigma(i)})^2=\prod_{i=1}^{k}(\alpha_i+\beta_{k-i+1})^2. \tag{8}$$

The expression $\max_{\substack{\sigma\in S_k\\ 1\le j\le k}}(\alpha_j-\beta_{\sigma(j)})^2\prod_{i\ne j}^{k}(\alpha_i+\beta_{\sigma(i)})^2$ can also be identified. Notice that for $i<j$ and $\sigma(i)<\sigma(j)$,

(a) if $\alpha_i-\beta_{\sigma(i)}\ge 0$, then $\alpha_i-\beta_{\sigma(j)}\ge 0$ and

$$(\alpha_i-\beta_{\sigma(i)})(\alpha_j+\beta_{\sigma(j)})-(\alpha_i-\beta_{\sigma(j)})(\alpha_j+\beta_{\sigma(i)})=-(\alpha_i+\alpha_j)(\beta_{\sigma(i)}-\beta_{\sigma(j)})\le 0.$$

(b) if $\alpha_i-\beta_{\sigma(i)}\le 0$, then $\alpha_j-\beta_{\sigma(i)}\le 0$ and

$$(\alpha_i-\beta_{\sigma(i)})(\alpha_j+\beta_{\sigma(j)})-(\alpha_i+\beta_{\sigma(j)})(\alpha_j-\beta_{\sigma(i)})=(\alpha_i-\alpha_j)(\beta_{\sigma(i)}-\beta_{\sigma(j)})\ge 0.$$

So

$$\max_{\substack{\sigma\in S_k\\ 1\le j\le k}}(\alpha_j-\beta_{\sigma(j)})^2\prod_{i\ne j}^{k}(\alpha_i+\beta_{\sigma(i)})^2=\max_{1\le j\le k}(\alpha_j-\beta_{k-j+1})^2\prod_{i\ne j}(\alpha_i+\beta_{k-i+1})^2$$

Since $[\alpha_k,\alpha_1]\cap[\beta_k,\beta_1]=\phi$, $\alpha_j+\beta_{k-j+1}>0$ for all $j$, it can be expressed as $\max_{1\le j\le k}\frac{(\alpha_j-\beta_{k-j+1})^2}{(\alpha_j+\beta_{k-j+1})^2}\prod_{i=1}^k(\alpha_i+\beta_{k-i+1})^2$.

The minimum is attained when most subtracted terms are present, i.e.,

Case 1. $k$ is even:

(i) if $\delta=1$, then for some $\sigma\in S_k$

$$B_0=\mathrm{diag}\,(-\beta_k,\dots,-\beta_1)\otimes J$$

yields the minimum $\prod_{i=1}^k(\alpha_i-\beta_{k-i+1})^2$. It is because

$$\min_{\sigma\in S_k}\prod_{i=1}^k(\alpha_i-\beta_{\sigma(i)})^2=\prod_{i=1}^k(\alpha_i-\beta_{k-i+1})^2 \tag{9}$$

since for definiteness we may assume that $\alpha_k>\beta_1$ and for $i<j$ and $\sigma(i)<\sigma(j)$,

$$(\alpha_i-\beta_{\sigma(i)})(\alpha_j-\beta_{\sigma(j)})-(\alpha_i-\beta_{\sigma(j)})(\alpha_j-\beta_{\sigma(i)})=(\alpha_i-\alpha_j)(\beta_{\sigma(i)}-\beta_{\sigma(j)})\ge 0$$

for each $\sigma\in S_k$. Also see [6, Theorem 1].

(ii) if $\delta=-1$, then for some $j$ and some $\sigma\in S_k$

$$B_0=\mathrm{diag}\,(-\beta_{\sigma(1)},\dots,\beta_{\sigma(j)},\dots,-\beta_{\sigma(k)})\otimes J$$

yields $\min_{\substack{\sigma\in S_k\\1\le j\le k}}(\alpha_j+\beta_{\sigma(j)})^2\prod_{i\ne j}(\alpha_i-\beta_{\sigma(i)})^2$. Since $[\alpha_k,\alpha_1]\cap[\beta_k,\beta_1]=\phi$, $(\alpha_j-\beta_{k-j+1})^2>0$ for all $j$. For definiteness, assume $\alpha_k>\beta_1$. Then

$$(\alpha_i+\beta_{\sigma(i)})(\alpha_j-\beta_{\sigma(j)})-(\alpha_i+\beta_{\sigma(j)})(\alpha_j-\beta_{\sigma(i)})=(\alpha_i+\alpha_j)(\beta_{\sigma(i)}-\beta_{\sigma(j)})\ge 0.$$

So

$$\min_{\substack{\sigma\in S_k\\1\le j\le k}}(\alpha_j+\beta_{\sigma(j)})^2\prod_{i\ne j}^k(\alpha_i-\beta_{\sigma(i)})^2=\min_{1\le j\le k}(\alpha_j+\beta_{k-j+1})^2\prod_{i\ne j}^k(\alpha_i-\beta_{k-i+1})^2 \tag{10}$$

which can be expressed as $\min_{1\le j\le k}\frac{(\alpha_j+\beta_{k-j+1})^2}{(\alpha_j-\beta_{k-j+1})^2}\prod_{i=1}^k(\alpha_i-\beta_{k-i+1})^2$. It is attained at one of the

$$B_0=\mathrm{diag}\,(-\beta_k,\dots,\beta_{k-j+1},\dots,-\beta_1)\otimes J,\quad j=1,\dots,k$$

Case 2. $k$ is odd:

(i) if $\delta=1$, one of the

$$B_0=\mathrm{diag}\,(-\beta_k,\dots,\beta_{k-j+1},\dots,-\beta_1)\otimes J,\quad j=1,\dots,k$$

yields $\min_{1\le j\le k}(\alpha_j+\beta_{k-j+1})^2\prod_{i\ne j}^k(\alpha_i-\beta_{k-i+1})^2$.

(ii) if $\delta = -1$, then

$$B_0 = \operatorname{diag}(-\beta_k, \ldots, -\beta_1) \otimes J$$

yields $\prod_{i=1}^k (\alpha_i - \beta_{k-i+1})^2$.

(b) Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$. The case $k = 1$ is trivial. When $k \geq 2$, by Lemma 2.2, $0 \in D_0(A, B)$ and hence $m_0 = 0$. For the maximum, by continuity argument and the fact that the set of singular skew matrices in $\mathbb{R}_{2k\times 2k}$ is of measure zero, we may assume that (4) holds and that $A + B$ is nonsingular. So $D_0(A, B) \neq \{0\}$ and thus the maximizing matrix $C_0$ in (5) is nonsingular. Then follow the argument in the proof of (a). □

**Corollary 2.4.** (a) When $\delta_A\delta_B = 1$, all matrices in $\Delta_0(A, B)$ are singular if and only if $\operatorname{rank} A + \operatorname{rank} B < 2k$.

(b) When $\delta_A\delta_B = -1$, all matrices in $\Delta_0(A, B)$ are singular if and only if $\operatorname{rank} A + \operatorname{rank} B < 2k$ or $\alpha_i = \beta_i = c > 0$ for all $i = 1, \ldots, k$.

*Proof.* (a) Suppose $\delta_A\delta_B = 1$. Then $D_0(A, B) = 0$ if and only if $M_0 = \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2 = 0$ from Theorem 2.3, i.e., $\alpha_i = \beta_{k-i+1} = 0$ for some $i = 1, \ldots, k$. In other words, $\operatorname{rank} A + \operatorname{rank} B < 2k$.

(b) Suppose $\delta_A\delta_B = -1$. Then $D_0(A, B) = 0$ if and only if $M_0 = \max_{1\leq j\leq k}(\alpha_j - \beta_{k-j+1})^2 \prod_{i\neq j}^k (\alpha_i + \beta_{k-i+1})^2 = 0$, i.e., either $\alpha_j - \beta_{k-j+1} = 0$ for all $j = 1, \ldots, k$, or $\alpha_i = \beta_{k-i+1} = 0$ for some $i = 1, \ldots, k$. The first amounts to $\alpha_i = \beta_i = c$, a constant, for all $i = 1, \ldots, k$. □

**Theorem 2.5.** Let $A, B \in \mathbb{R}_{2k\times 2k}$ be real skew symmetric with singular values $\alpha_1 \geq \alpha_1 \geq \alpha_2 \geq \alpha_2 \geq \cdots \geq \alpha_k \geq \alpha_k$ and $\beta_1 \geq \beta_1 \geq \beta_2 \geq \beta_2 \geq \cdots \geq \beta_k \geq \beta_k$ respectively.

(I) If $k = 1$, then $D(A, B) = \{(\alpha_1 - \beta_1)^2, (\alpha_1 + \beta_1)^2\}$.

(II) Let $k \geq 2$. Then $D(A, B) = [m, M]$, where

$$M = \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2$$

and

$$m = \begin{cases} \prod_{i=1}^k (\alpha_i - \beta_{k-i+1})^2 & \text{if} \quad [\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi \\ 0 & \text{if} \quad [\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi, \end{cases}$$

*Proof.* (I) is trivial.
(II) By Lemma 2.1(b)

$$D(A, B) = [m_0(A, B), M_0(A, B)] \cup [m_0(A_D, B), M_0(A_D, B)],$$

where

$$M_0(A,B) := \begin{cases} \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2 & \text{if } \delta_A\delta_B = 1 \\ \max_{1\le j\le k}(\alpha_j - \beta_{k-j+1})^2 \prod_{i\ne j}^k (\alpha_i + \beta_{k-i+1})^2 & \text{if } \delta_A\delta_B = -1 \end{cases}$$

and

$$M_0(A_D,B) := \begin{cases} \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2 & \text{if } \delta_A\delta_B = -1 \\ \max_{1\le j\le k}(\alpha_j - \beta_{k-j+1})^2 \prod_{i\ne j}^k (\alpha_i + \beta_{k-i+1})^2 & \text{if } \delta_A\delta_B = 1 \end{cases}$$

since $\delta_{A_D} = -\delta_A$, according to Theorem 2.3. So $M$ is the maximum of $M_0(A,B)$ and $M_0(A_D,B)$ and is $\prod_{i=1}^k(\alpha_i + \beta_{k-i+1})^2$.

(i) If $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \ne \phi$, then $m_0(A,B) = m_0(A_D,B) = 0$ by Theorem 2.3. So $m = 0$ and $D(A,B)$ is the interval $[0, M]$.

(ii) If $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$, we have $m = \min\{m_0(A,B), m_0(A_D,B)\} = \prod_{i=1}^k (\alpha_i - \beta_{k-i+1})^2$, again by Theorem 2.3. It remains to show that $D(A,B)$ is an interval, i.e., $D_0(A,B) \cap D_0(A_D,B) \ne \phi$. Let

$$\begin{aligned} L &:= \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2 \\ \ell &:= \max_{1\le j\le k} (\alpha_j - \beta_{k-j+1})^2 \prod_{i\ne j}^k (\alpha_i + \beta_{k-i+1})^2 \\ s &:= \min_{1\le j\le k} (\alpha_j + \beta_{k-j+1})^2 \prod_{i\ne j}^k (\alpha_i - \beta_{k-i+1})^2 \\ S &:= \prod_{i=1}^k (\alpha_i - \beta_{k-i+1})^2 \end{aligned}$$

Clearly $S \le s \le \ell \le L$. To show that $D_0(A,B) \cap D_0(A_D,B) \ne \phi$ it suffices to consider $\delta_A\delta_B = 1$ by symmetry. Suppose $\delta_A\delta_B = 1$. By Theorem 2.3, when $k$ is even $D_0(A_D,B) = [s,\ell] \subseteq [S,L] = D_0(A,B)$; when $k$ is odd $D_0(A,B) = [s,L]$ and $D_0(A_D,B) = [S,\ell]$. Hence $D(A,B)$ is an interval. □

**Example 2.6.** Let $k = 2$ and $A = \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$. Since $[1,3] \cap [2,4] \ne \phi$, $S(A,B) = [0, 625]$ by Theorem 2.5. Since $k = 2$,

$$\begin{aligned} S(A,B) &\subset \{\det(UAU^{-1} + VBV^{-1}) : U, V \in \mathrm{U}(4)\} \\ &= \{\det(U(iA)U^{-1} + V(iB)V^{-1}) : U, V \in \mathrm{U}(4)\} \\ &= H(iA, iB). \end{aligned}$$

The eigenvalues of $iA$ are $3, 1, -1, -3$ (denoted by $\alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4$), and those of $iB$ are $4, 2, -2, -4$ (denoted by $\beta'_1, \beta'_2, \beta'_3, \beta'_4$). None of $\prod_{i=1}^4(\alpha'_i + \beta'_{\sigma(i)})$, $\sigma \in S_4$, is zero. Indeed, if $\tau = (13) \in S_4$ denotes the transposition, then $\beta'_\tau = (-2, 2, 4, 4)$. So $\prod_{i=1}^4(\alpha'_i + \beta'_{\tau(i)}) = -63 \notin S(A, B)$.

Similarly if we consider $A = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 3 \\ -3 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Since $[3, 4] \cap [1, 2] = \phi$, $S(A, B) = [3, 625]$ by Theorem 2.5. $\prod_{i=1}^4(\alpha'_i + \beta'_{\tau(i)}) = -24 \notin S(A, B)$. So the lower bound of Theorem 1.1 is in general not included in $S(A, B)$.

## 3. Pfaffian and sum of orbits

We consider

$$P(A, B) := \{\text{Pf}\,(UAU^{-1} + VBV^{-1}) : U, V \in \text{O}(2k)\}.$$

and

$$P_0(A, B) := \{\text{Pf}\,(UAU^{-1} + VBV^{-1}) : U, V \in \text{SO}(2k)\}.$$

Clearly $P_0(A, B)$ is an interval.

**Theorem 3.1.** Let $A, B \in \mathbb{R}_{2k\times 2k}$ be real skew symmetric with singular values $\alpha_1 \geq \alpha_1 \geq \cdots \geq \alpha_k \geq \alpha_k \geq 0$ and $\beta_1 \geq \beta_1 \geq \cdots \geq \beta_k \geq \beta_k \geq 0$ respectively.

(a) If $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$, then $P(A, B) = [-P, P]$, where $P = \prod_{i=1}^k(\alpha_i + \beta_{k-i+1})$.

(b) If $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$, then $P(A, B) = [-P, -p] \cup [p, P]$ where $p = \prod_{i=1}^k |\alpha_i - \beta_{k-i+1}|$.

*Proof.* Since

$$P(A, B) = P_0(A, B) \cup P_0(A_D, B) \cup P_0(A, B_D) \cup P_0(A_D, B_D)$$

and

$$P_0(A_D, B_D) = -P_0(A, B), \quad P_0(A, B_D) = -P_0(A_D, B), \tag{11}$$

$P(A, B)$ is symmetric about the origin. Since $\det A = (\text{Pf}\,A)^2$, using Theorem 2.5 we have the desired result. ☐

**Theorem 3.2.** Let $A, B \in \mathbb{R}_{2k\times 2k}$ be real skew symmetric with singular values $\alpha_1 \geq \alpha_1 \geq \cdots \geq \alpha_k \geq \alpha_k \geq 0$ and $\beta_1 \geq \beta_1 \geq \cdots \geq \beta_k \geq \beta_k \geq 0$ respectively. Let $\delta_A$ and $\delta_B$ be the signs of the Pfaffians of $A$ and $B$, respectively.

(I) If $k = 1$, then $P_0(A, B) = \{s\}$, where

$$s = \begin{cases} \alpha_1 + \beta_1 & \text{if } \delta_A = \delta_B = 1 \\ -(\alpha_1 + \beta_1) & \text{if } \delta_A = \delta_B = -1 \\ \alpha_1 - \beta_1 & \text{if } \delta_A = 1, \delta_B = -1 \\ -\alpha_1 + \beta_1 & \text{if } \delta_A = -1, \delta_B = 1 \end{cases}$$

(II) Let $k \geq 2$. Set $P_0(A, B) = [p_0, P_0]$.

(a) Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$.

(1) If $\delta_A = \delta_B = 1$, then $P_0 = \prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})$ and

$$p_0 = \begin{cases} \prod_{i=1}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ even} \\ \min_{1\leq j\leq k}(\alpha_j + \beta_{k-j+1}) \prod_{i\neq j}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ odd} \end{cases}$$

(2) If $\delta_A = \delta_B = -1$, then

$$P_0 = \begin{cases} -\prod_{i=1}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ even} \\ -\min_{\substack{\sigma\in S_k \\ 1\leq j\leq k}} (\alpha_j + \beta_{\sigma(j)}) \prod_{i\neq j} |\alpha_i - \beta_{\sigma(i)}| & \text{if } k \text{ odd} \end{cases}$$

and $p_0 = -\prod_{i=1}^{k}(\alpha_i + \beta_{k-i+1})$.

(3) If $\delta_A = 1$ and $\delta_B = -1$, and

(i) if $\alpha_k > \beta_1$, then

$$P_0 = \max_{1\leq j\leq k} |\alpha_j - \beta_{k-j+1}| \prod_{i\neq j}^{k} (\alpha_i + \beta_{k-i+1})$$

and

$$p_0 = \begin{cases} \min_{1\leq j\leq k}(\alpha_j + \beta_{k-j+1}) \prod_{i\neq j}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ is even} \\ \prod_{i=1}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ is odd} \end{cases}$$

(ii) if $\beta_k > \alpha_1$, then

$$P_0 = \begin{cases} -\min_{1\leq j\leq k}(\alpha_j + \beta_{k-j+1}) \prod_{i\neq j}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ is even} \\ -\prod_{i=1}^{k} |\alpha_i - \beta_{k-i+1}| & \text{if } k \text{ is odd} \end{cases}$$

and

$$p_0 = -\max_{1\leq j\leq k} |\alpha_j - \beta_{k-j+1}| \prod_{i\neq j}^{k} (\alpha_i + \beta_{k-i+1}) < 0$$

(4) If $\delta_A = -1$ and $\delta_B = 1$, and

(i) if $\alpha_k > \beta_1$, then

$$P_0 = \begin{cases} -\min_{1\le j\le k}(\alpha_j+\beta_{k-j+1})\prod_{i\neq j}^k|\alpha_i-\beta_{k-i+1}| & \text{if } k \text{ is even} \\ -\prod_{i=1}^k|\alpha_i-\beta_{k-i+1}| & \text{if } k \text{ is odd} \end{cases}$$

and

$$p_0 = -\max_{1\le j\le k}|\alpha_j-\beta_{k-j+1}|\prod_{i\neq j}^k(\alpha_i+\beta_{k-i+1}) < 0$$

(ii) if $\beta_k > \alpha_1$, then

$$P_0 = \max_{1\le j\le k}|\alpha_j-\beta_{k-j+1}|\prod_{i\neq j}^k(\alpha_i+\beta_{k-i+1})$$

and

$$p_0 = \begin{cases} \min_{1\le j\le k}(\alpha_j+\beta_{k-j+1})\prod_{i\neq j}^k|\alpha_i-\beta_{k-i+1}| & \text{if } k \text{ is even} \\ \prod_{i=1}^k|\alpha_i-\beta_{k-i+1}| & \text{if } k \text{ is odd} \end{cases}$$

(b) Suppose $[\alpha_k,\alpha_1]\cap[\beta_k,\beta_1]\neq\phi$.

(1) If $\delta_A=\delta_B=1$, then $P_0(A,B)=[a,\prod_{i=1}^k(\alpha_i+\beta_{k-i+1})]$ for some $-\prod_{i=1}^k(\alpha_i+\beta_{k-i+1})\le a\le 0$.

(2) If $\delta_A=\delta_B=-1$, then $P_0(A,B)=[-\prod_{i=1}^k(\alpha_i+\beta_{k-i+1}),b]$ for some $0\le b\le\prod_{i=1}^k(\alpha_i+\beta_{k-i+1})$.

(3) If $\delta_A=1$, $\delta_B=-1$, or $\delta_A=-1$, $\delta_B=1$, then $P_0(A,B)$ is either

$$[c,\max_{1\le j\le k}|\alpha_j-\beta_{k-j+1}|\prod_{i\neq j}^k(\alpha_i+\beta_{k-i+1})]$$

for some $-\max_{1\le j\le k}|\alpha_j-\beta_{k-j+1}|\prod_{i\neq j}^k(\alpha_i+\beta_{k-i+1})\le c\le 0$, or

$$[-\max_{1\le j\le k}|\alpha_j-\beta_{k-j+1}|\prod_{i\neq j}^k(\alpha_i+\beta_{k-i+1}),d]$$

for some $0\le d\le\max_{1\le j\le k}|\alpha_j-\beta_{k-j+1}|\prod_{i\neq j}^k(\alpha_i+\beta_{k-i+1})$.

*Proof.* (I) is trivial.

(II) Clearly $P_0(A,B)$ is a closed interval. Since $D_0(A,B)=[m_0,M_0]$, where $m_0, M_0\ge 0$ are given in Theorem 2.3 and $\det A=(\mathrm{Pf}\,A)^2$, $P_0(A,B)$ is either $[m_0^{1/2},M_0^{1/2}]$ or $[-M_0^{1/2},-m_0^{1/2}]$. It is also clear that

$$P_0(A_D,B_D)=-P_0(A,B),\quad P_0(A,B_D)=-P_0(A_D,B).$$

(a) Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi$. By Lemma 2.2, $0 \notin P_0(A, B)$ so that either $P_0(A, B) \subseteq \mathbb{R}_+$ or $P_0(A, B) \subseteq \mathbb{R}_-$.

(1) If $\delta_A = \delta_B = 1$, then clearly $0 < \prod_{i=1}^k (\alpha_i + \beta_{k-i+1}) \in P_0(A, B)$. So $P_0(A, B) = [m_0^{1/2}, M_0^{1/2}]$.
(2) If $\delta_A = \delta_B = -1$, then clearly $0 > -\prod_{i=1}^k (\alpha_i + \beta_{k-i+1}) \in P_0(A, B)$. So $P_0(A, B) = [-M_0^{1/2}, -m_0^{1/2}]$.
(3) Suppose $\delta_A = 1$ and $\delta_B = -1$.
(i) if $\alpha_k > \beta_1$, then $0 < \max_{1\le j\le k}(\alpha_j - \beta_{k-j+1}) \prod_{i\neq j}^k (\alpha_i + \beta_{k-i+1}) \in P_0(A, B)$. So $P_0(A, B) = [m_0^{1/2}, M_0^{1/2}]$.
(ii) if $\beta_k > \alpha_1$, then $0 > \max_{1\le j\le k}(\alpha_j - \beta_{k-j+1}) \prod_{i\neq j}^k (\alpha_i + \beta_{k-i+1}) \in P_0(A, B)$. So $P_0(A, B) = [-M_0^{1/2}, -m_0^{1/2}]$.

(4) Similar.

(b) Suppose $[\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi$. Then $0 \in P_0(A, B)$ by Theorem 2.3. (1) and (2) follow from Theorem 2.3. (3) Clearly $P_0(A, B)$ contains $\max_{1\le j\le k} |\alpha_j - \beta_{k-j+1}| \prod_{i\neq j}^k (\alpha_i + \beta_{k-i+1})$ or its negative by Theorem 2.3. □

We were not able to obtain $a, b, c, d$.

## 4. Complex case

Given complex skew symmetric matrices $A, B \in \mathbb{C}_{2k\times 2k}$ with singular values $\alpha_1 \ge \alpha_1 \ge \cdots \ge \alpha_k \ge \alpha_k \ge 0$ and $\beta_1 \ge \beta_1 \ge \cdots \ge \beta_k \ge \beta_k \ge 0$ respectively. Consider the complex analog

$$\Delta_{\mathbb{C}}(A, B) := \{UAU^T + VBV^T : U, V \in \mathrm{U}(2k)\},$$

$$D_{\mathbb{C}}(A, B) := \{\det M : M \in \Delta_{\mathbb{C}}(A, B)\}$$

and

$$P_{\mathbb{C}}(A, B) := \{\mathrm{Pf}\, M : M \in \Delta_{\mathbb{C}}(A, B)\}.$$

**Theorem 4.1.** Let $A, B \in \mathbb{C}_{2k\times 2k}$ be complex skew symmetric with singular values $\alpha_1 \ge \alpha_1 \ge \alpha_2 \ge \alpha_2 \ge \cdots \ge \alpha_k \ge \alpha_k$ and $\beta_1 \ge \beta_1 \ge \beta_2 \ge \beta_2 \ge \cdots \ge \beta_k \ge \beta_k$ respectively. Then

(1) $D_{\mathbb{C}}(A, B)$ is an annulus of inner radius $r$ and outer radius $R$, where

$$R = \prod_{i=1}^k (\alpha_i + \beta_{k-i+1})^2$$

and

$$r = \begin{cases} \prod_{i=1}^{k}(\alpha_i - \beta_{k-i+1})^2 & \text{if} \quad [\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] = \phi \\ 0 & \text{if} \quad [\alpha_k, \alpha_1] \cap [\beta_k, \beta_1] \neq \phi \end{cases}$$

(2) $P_{\mathbb{C}}(A, B)$ is an annulus of inner radius $\sqrt{r}$ and outer radius $\sqrt{R}$.

*Proof.* Clearly $D_{\mathbb{C}}(A, B)$ admits circular symmetry and by the connectedness of $\mathrm{U}(n)$, $D_{\mathbb{C}}(A, B)$ is an annulus. By Autonne's decomposition [1] (also see [10]) we may assume that

$$A = \mathrm{diag}\,(\alpha_1, \ldots, \alpha_k) \otimes J, \quad B = \mathrm{diag}\,(\beta_1, \ldots, \beta_k) \otimes J.$$

The radii of $D_{\mathbb{C}}(A, B)$ can be deduced from Theorem 2.5 and [6, Theorem 1] since $\alpha$'s and $\beta$'s are singular values of $A$ and $B$ respectively. $P_{\mathbb{C}}(A, B)$ also admits circular symmetry since

$$\begin{aligned} & e^{i\theta}\mathrm{Pf}\,(UAU^T + VBV^T) \\ = \;& \det(e^{i\theta/2k}I)\mathrm{Pf}\,(UAU^T + VBV^T) \\ = \;& \mathrm{Pf}\,(e^{i\theta/2k}UA(e^{i\theta/2k}U)^T + e^{i\theta/2k}VB(e^{i\theta/2k}V)^T) \end{aligned}$$

using $\mathrm{Pf}\,(PAP^T) = (\det P)\,\mathrm{Pf}\,A$. □

We remark that circular symmetry does not exist if $\mathrm{U}(2k)$ is replaced by $\mathrm{SU}(2k)$.

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