3.1

Remarks: The notion of solvability imitates the corresponding notion in group theory searching a proof of the general unsolvability of quintic and higher equations, finally realized by Galois theory.

Questions: 1. Show that the derived series is descending, i.e., $L^{i+1} \subset L^i$.
2. Show that $t(n, F)$ is solvable.
3. Show that the radical of $L$ is the smallest ideal $I$ such that $\text{Rad } L/I = 0$.
4. Show that $L/\text{Rad } L$ is semisimple.

3.2

Remarks: Another proof of Lemma 3.2 If $x \in \mathfrak{gl}(V)$ is nilpotent, say, $x^p = 0$, then $(\text{ad } x)^{2p-1} = 0$, i.e., $\text{ad } x$ is also nilpotent.

Proof. $(\text{ad } x)^n y$ is the sum of terms of the form $\pm x^i y x^j$ with $i + j = n$. □

Questions: 1. Show that $n(n, F)$ is a nilpotent algebra.
2. Show that the descending central series is descending, i.e., $L^{i+1} \subset L^i$.
3. Show that the center of a nonzero nilpotent $L$ is nonzero.
4. Is the term nilpotent algebra justified according to Lemma 3.2?
5. Show that abelian $\Rightarrow$ nilpotent $\Rightarrow$ solvable.
6. Show that $L$ is solvable if and only if $L^1 = [L, L]$ is nilpotent.
7. Given an ideal $I$ of $L$, what is the relation between $(L/I)^{(n)}$ and $L^{(n)}/I$?
Exercises on Section 3

2. Suppose that \( L \) is solvable. The derived series \( L = L^{(0)} \supseteq L^{(1)} \supseteq \cdots \supseteq L^{(n)} = 0 \) satisfies the properties. We know from 2.1 that \( L^{(i+1)} = [L^{(i)}, L^{(i)}] \) is an ideal of \( L^{(i)} \). It remains to show that \( L^{(i)}/L^{(i+1)} \) is abelian.

For \( x, y \in L^{(i)} \), \([x, y] \in L^{(i+1)} \)

\[
[x + L^{(i+1)}, y + L^{(i+1)}] \subseteq [x, y] + [x, L^{(i+1)}] + [y, L^{(i+1)}] + [L^{(i+1)}, L^{(i+1)}] \subseteq L^{(i+1)},
\]
i.e., \( L^{(i)}/L^{(i+1)} \) is abelian.

Conversely suppose that \( L = L_0 \supseteq L_1 \supseteq L_2 \supseteq \cdots \supseteq L_k = 0 \) such that \( L_{i+1} \) is an ideal of \( L_i \) and \( L_i/L_{i+1} \) is abelian for all \( i \). We will show that all \( L_i \) are solvable and use backward induction (clearly \( L_k = 0 \) is solvable). Clearly \( L_i/L_{i+1} \) is solvable for all \( i \). Now \( L_{k-1} \supseteq L_{k-1}/L_k \) is solvable so that \( L_{k-1} \) is a solvable ideal of \( L_{k-2} \). Now \( L_{k-2}/L_{k-1} \) is abelian and thus solvable. By Proposition 3.1(b), \( L_{k-2} \) is solvable. By induction argument we have the desired result.

Remark: We may add an additional condition \( \dim L_i/L_{i+1} = 1 \) for all \( i \).

\[\]

Proof. From the derived series, we interpolate subspaces (how?) so that \( \dim L_i/L_{i+1} = 1 \).

Conversely choose \( x_i \) so that \( L_i = \mathbb{F}x_i + L_{i+1} \) (direct sum). We show by induction that \( L^{(i)} \subseteq L_i \) so that \( L^{(k)} = 0 \). In fact \( L = L_0 \). If \( L^{(i)} \subseteq L_i \), then

\[
L^{(i+1)} = [L^{(i)}, L^{(i)}] \subseteq [\mathbb{F}x_i + L_{i+1}, \mathbb{F}x_i + L_{i+1}] \subseteq [\mathbb{F}x_i, L_{i+1}] + [L_{i+1}, L_{i+1}] \subseteq L_{i+1}.
\]

4. Recall \( L^{(i)} = [L^{(i-1)}, L^{(i-1)}] \) (p.10). Notice that \( \text{ad } L \subseteq \mathfrak{gl}(L) \) is a Lie algebra and the derived series is given by

\[
(\text{ad } L)^{(i)} = [(\text{ad } L)^{(i-1)}, (\text{ad } L)^{(i-1)}].
\]

Since \( \text{ad } : L \to \mathfrak{gl}(L) \) is a homomorphism (p.8),

\[
(\text{ad } L)^{(1)} := [\text{ad } L, \text{ad } L] = \text{ad } [L, L] = \text{ad } L^{(1)}.
\]

By induction

\[
(\text{ad } L)^{(i)} = \text{ad } (L^{(i)}),
\]
i.e., \( \text{ad } \) and \((\cdot)^{(i)}\) commute. Clearly if \( L \) is solvable, so is \( \text{ad } L \) because \( \text{ad } 0 = 0 \), or simply by Proposition 3.1 (a). Conversely \( \text{ad } L \) solvable implies \( \text{ad } L^{(n)} = 0 \) for some \( n \). So \( L^{(n)} \subseteq Z \) is abelian since \( Z = \text{Ker } \text{ad} \) (p.8). Then \( L^{(n+1)} = 0 \), i.e., \( L \) is solvable.

Similar for nilpotency. First use Proposition 3.2(a). Then show by induction that

\[
(\text{ad } L)^{(i)} = \text{ad } (L^{(i)}),
\]
i.e., \( \text{ad } \) and \((\cdot)^{(i)}\) commute. So \( \text{ad } L \) nilpotent implies \( \text{ad } (L^n) = 0 \) for some \( n \), i.e., \( L^{n+1} = [L^n, L] = 0 \).

5. (a) \( L^{(1)} = [L, L] \subseteq \text{span } x \). So \( L^{(2)} = 0 \) and hence \( L \) is solvable. Moreover \( x \in L^i \) for all \( i \) so \( L \) is not nilpotent.

(b) Consider \( \text{ad } (x + y) \) satisfying

\[
\text{ad } (x + y)(x) = -z, \quad \text{ad } (x + y)(y) = z, \quad \text{ad } (x + y)(z) = y,
\]

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so that the matrix of \( \text{ad} (x + y) \) is \[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
-1 & 1 & 0
\end{pmatrix}
\]
Notice that \( (\text{ad} (x + y))^3 \neq 0 \) and index of nilpotency is always less than or equal to the dimension of the matrix, i.e., 3. So by Engel’s theorem, \( L \) is not nilpotent.

6. The sum of two ideals is still an ideal (p.6). So we only need to show that \( I + J \) is nilpotent if \( I \) and \( J \) are nilpotent ideal. Each element of \( (I + J)^n \) is of the form

\[
[x_1 + y_1, [x_2 + y_2, \cdots [x_n + y_n, x_{n+1}, y_{n+1}]], x_i \in I, y_i \in J,
\]
which is a sum of terms of the form

\[
z = [z_1, [z_2, \cdots [z_n, z_{n+1}] \cdots]],
\]
where \( z_i \) is either in \( I \) or \( J \). Suppose that \( z \) has \( m z_i \) in \( I \), i.e., \( n - m z_i \) in \( J \). Then \( z \in I^{m-1} \) or equivalently \( z \in J^{n-m-1} \). Choose sufficiently large \( n \), \( z = 0 \), since both \( I \) and \( J \) are nilpotent.

(The statement is analogous to Proposition 3.1) Thus \( L \) possesses a unique maximal (set inclusion) nilpotent ideal.

(a) Refer to the example on p.5, i.e., \([x, y] = x\). The algebra \( L \) is not nilpotent by Exercise 3.5. Hence the span of \( x \) is the maximal nilpotent ideal.

(b) Refer to Exercise 1.2. The algebra \( L \) is not nilpotent. Hence the span of \( x \) and \( y \) is the maximal nilpotent ideal.

8. (An ideal \( I \) of codimension 1 means that \( \dim I = \dim L - 1 \)) Let \( K \) be a maximal proper subalgebra of \( L \). By Exercise 3.7, \( K \subseteq N_L(K) \) where \( N_L(K) \) is a subalgebra of \( L \). By the maximality of \( K \), \( L = N_L(K) \), i.e., \( K \) is an ideal, so that \( L/K \) is well-defined. If \( \dim L/K \) were greater than 1, then the inverse image (with respect to the natural map \( L \to L/K \)) in \( L \) of a one dimensional subalgebra of \( L/K \) (which always exists) would be a proper subalgebra properly containing \( K \), a contradiction (see p.13). Therefore \( K \) has codimension one.