

Lie Algebra Math 7970 Solutions and Remarks

3.1

Remarks: The notion of **solvability** imitates the corresponding notion in group theory searching a proof of the general unsolvability of quintic and higher equations, finally realized by Galois theory.

- Questions:
1. Show that the derived series is descending, i.e., $L^{i+1} \subset L^i$.
 2. Show that $\mathfrak{t}(n, \mathbb{F})$ is solvable.
 3. Show that the radical of L is the smallest ideal I such that $\text{Rad } L/I = 0$.
 4. Show that $L/\text{Rad } L$ is semisimple.

3.2

Remarks: Another proof of Lemma 3.2 If $x \in \mathfrak{g}(V)$ is nilpotent, say, $x^p = 0$, then $(\text{ad } x)^{2p-1} = 0$, i.e., $\text{ad } x$ is also nilpotent.

Proof. $(\text{ad } x)^n y$ is the sum of terms of the form $\pm x^i y x^j$ with $i + j = n$. □

- Questions:
1. Show that $\mathfrak{n}(n, \mathbb{F})$ is a nilpotent algebra.
 2. Show that the descending central series is descending, i.e., $L^{i+1} \subset L^i$.
 3. Show that the center of a nonzero nilpotent L is nonzero.
 4. Is the term nilpotent algebra justified according to Lemma 3.2?
 5. Show that abelian \Rightarrow nilpotent \Rightarrow solvable.
 6. Show that L is solvable if and only if $L^1 = [L, L]$ is nilpotent.
 7. Given an ideal I of L , what is the relation between $(L/I)^{(n)}$ and $L^{(n)}/I$?

Exercises on Section 3

2. Suppose that L is solvable. The derived series $L = L^{(0)} \supset L^{(1)} \supset \dots \supset L^{(n)} = 0$ satisfies the properties. We know from 2.1 that $L^{(i+1)} = [L^{(i)}, L^{(i)}]$ is an ideal of $L^{(i)}$. It remains to show that $L^{(i)}/L^{(i+1)}$ is abelian. For $x, y \in L^{(i)}$, $[x, y] \in L^{(i+1)}$

$$[x + L^{(i+1)}, y + L^{(i+1)}] \subset [x, y] + [x, L^{(i+1)}] + [y, L^{(i+1)}] + [L^{(i+1)}, L^{(i+1)}] \subset L^{(i+1)},$$

i.e., $L^{(i)}/L^{(i+1)}$ is abelian.

Conversely suppose that $L = L_0 \supset L_1 \supset L_2 \supset \dots \supset L_k = 0$ such that L_{i+1} is an ideal of L_i and L_i/L_{i+1} is abelian for all i . We will show that all L_i are solvable and use backward induction (clearly $L_k = 0$ is solvable). Clearly L_i/L_{i+1} is solvable for all i . Now $L_{k-1} \cong L_{k-1}/L_k$ is solvable so that L_{k-1} is a solvable ideal of L_{k-2} . Now L_{k-2}/L_{k-1} is abelian and thus solvable. By Proposition 3.1(b), L_{k-2} is solvable. By induction argument we have the desired result.

Remark: We may add an additional condition $\dim L_i/L_{i+1} = 1$ for all i .

Proof. From the derived series, we interpolate subspaces (how?) so that $\dim L_i/L_{i+1} = 1$.

Conversely choose x_i so that $L_i = \mathbb{F}x_i + L_{i+1}$ (direct sum). We show by induction that $L^{(i)} \subset L_i$ so that $L^{(k)} = 0$. In fact $L = L_0$. If $L^{(i)} \subset L_i$, then

$$L^{(i+1)} = [L^{(i)}, L^{(i)}] \subset [\mathbb{F}x_i + L_{i+1}, \mathbb{F}x_i + L_{i+1}] \subset [\mathbb{F}x_i, L_{i+1}] + [L_{i+1}, L_{i+1}] \subset L_{i+1}.$$

□

4. Recall $L^{(i)} = [L^{(i-1)}, L^{(i-1)}]$ (p.10). Notice that $\text{ad } L \subset \mathfrak{gl}(L)$ is a Lie algebra and the derived series is given by

$$(\text{ad } L)^{(i)} = [(\text{ad } L)^{(i-1)}, (\text{ad } L)^{(i-1)}].$$

Since $\text{ad} : L \rightarrow \mathfrak{gl}(L)$ is a homomorphism (p.8),

$$(\text{ad } L)^{(1)} := [\text{ad } L, \text{ad } L] = \text{ad } [L, L] = \text{ad } L^{(1)}.$$

By induction

$$(\text{ad } L)^{(i)} = \text{ad } (L^{(i)}),$$

i.e., ad and $(\cdot)^{(i)}$ commute. Clearly if L is solvable, so is $\text{ad } L$ because $\text{ad } 0 = 0$, or simply by Proposition 3.1 (a). Conversely $\text{ad } L$ solvable implies $\text{ad } L^{(n)} = 0$ for some n . So $L^{(n)} \subset Z$ is abelian since $Z = \text{Ker } \text{ad}$ (p.8). Then $L^{(n+1)} = 0$, i.e., L is solvable.

Similar for nilpotency. First use Proposition 3.2(a). Then show by induction that

$$(\text{ad } L)^i = \text{ad } (L^i),$$

i.e., ad and $(\cdot)^i$ commute. So $\text{ad } L$ nilpotent implies $\text{ad } (L^n) = 0$ for some n , i.e., $L^{n+1} = [L^n, L] = 0$.

5. (a) $L^{(1)} = [L, L] \subset \text{span } x$. So $L^{(2)} = 0$ and hence L is solvable. Moreover $x \in L^i$ for all i so L is not nilpotent.

(b) Consider $\text{ad } (x + y)$ satisfying

$$\text{ad } (x + y)(x) = -z, \quad \text{ad } (x + y)(y) = z, \quad \text{ad } (x + y)(z) = y,$$

so that the matrix of $\text{ad}(x+y)$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}$. Notice that $(\text{ad}(x+y))^3 \neq 0$ and index of nilpotency is always less than or equal to the dimension of the matrix, i.e., 3. So by Engel's theorem, L is not nilpotent.

6. The sum of two ideals is still an ideal (p.6). So we only need to show that $I+J$ is nilpotent if I and J are nilpotent ideal. Each element of $(I+J)^n$ is of the form

$$[x_1 + y_1, [x_2 + y_2, \dots [x_n + y_n, x_{n+1}, y_{n+1}] \dots]], \quad x_i \in I, y_i \in J,$$

which is a sum of terms of the form

$$z = [z_1, [z_2, \dots [z_n, z_{n+1}] \dots]],$$

where z_i is either in I or J . Suppose that z has m z_i in I , i.e., $n-m$ z_i in J . Then $z \in I^{m-1}$ or equivalently $z \in J^{n-m-1}$. Choose sufficiently large n , $z = 0$, since both I and J are nilpotent. (The statement is analogous to Proposition 3.1) Thus L possesses a unique maximal (set inclusion) nilpotent ideal.

- (a) Refer to the example on p.5, i.e., $[x, y] = x$. The algebra L is not nilpotent by Exercise 3.5. Hence the span of x is the maximal nilpotent ideal.
- (b) Refer to Exercise 1.2. The algebra L is not nilpotent. Hence the span of x and y is the maximal nilpotent ideal.

8. (An ideal I of codimension 1 means that $\dim I = \dim L - 1$) Let K be a maximal proper subalgebra of L . By Exercise 3.7, $K \subsetneq N_L(K)$ where $N_L(K)$ is a subalgebra of L . By the maximality of K , $L = N_L(K)$, i.e., K is an ideal, so that L/K is well-defined. If $\dim L/K$ were greater than 1, then the inverse image (with respect to the natural map $L \rightarrow L/K$) in L of a one dimensional subalgebra of L/K (which always exists) would be a proper subalgebra properly containing K , a contradiction (see p.13). Therefore K has codimension one.